Math 546, Final Exam, Fall, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

No calculators, cell phones, computers, notes, etc.

$\boxed{\text{Make your work correct, complete, and coherent.}}$

The exam is worth 100 points. Each problem is worth 10 points.

(1) State Cayley's theorem.

(2) State Lagrange's theorem.

(3) Is $\phi : \frac{\mathbb{Z}}{2\mathbb{Z}} \to \frac{\mathbb{Z}}{4\mathbb{Z}}$, which is given by $\phi(j + 2\mathbb{Z}) = j + 4\mathbb{Z}$, for $j \in \mathbb{Z}$, a group homomorphism? Explain thoroughly.

(4) Let a and b be elements of finite order in the group G. Does ab have to have finite order? Explain thoroughly.

(5) Let a and b be elements of finite order in the group G. Suppose ab has finite order. Does the order of ab have to equal the order of a times the order of b? Explain thoroughly.

(6) Let a and b be elements of finite order in the group G. List some conditions on a, b, and G which guarantee that the order of ab is equal the order of a times the order of b. Prove the resulting statement. (Make your statement work for as many situations as possible.)

(7) List five groups of order eight. Explain why no two groups on your list are isomorphic.

(8) Let U be the unit circle subgroup of $(\mathbb{C} \setminus \{0\}, \times)$. In other words,

$$U = \{a + bi \mid a \text{ and } b \text{ are real numbers and } a^2 + b^2 = 1\}.$$

Let U_2 be the subgroup $\{1, -1\}$ of U. What familiar group is isomorphic to $\frac{U}{U_2}$? Prove your assertion.

(9) Let m and n be relatively prime positive integers. Prove that the groups $\frac{\mathbb{Z}}{n\mathbb{Z}} \oplus \frac{\mathbb{Z}}{m\mathbb{Z}}$ and $\frac{\mathbb{Z}}{nm\mathbb{Z}}$ are isomorphic.

(10) Give an example of integers n and m with $\frac{\mathbb{Z}}{n\mathbb{Z}} \oplus \frac{\mathbb{Z}}{m\mathbb{Z}}$ and $\frac{\mathbb{Z}}{nm\mathbb{Z}}$ not isomorphic. Explain thoroughly.