Math 546, Final Exam , Fall 2004

The exam is worth 100 points.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

I will grade the exams on Saturday. When I finish, I will e-mail your grade to you.

I will post the solutions on my website when the exam is finished.

1. (7 points) STATE and PROVE Cayley's Theorem.

2. (7 points) Apply the proof of Cayley's Theorem to the element $(1,2,3)$ of the group

$$A_4 = \{(1),(1,2,3),(1,3,2),(1,2,4),(1,4,2),(1,3,4),(1,4,3),(2,3,4),(2,4,3),$$

$$(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}.$$

   (Number the elements of A_4 using the order I in which I listed the elements.) What do you get?

3. (7 points) Let $\varphi\colon G \to G'$ be a group homomorphism. Prove that φ is one-to-one if and only if the kernel of φ is $\{\mathrm{id}\}$.

4. (7 points) Give an example of a non-abelian group of order 16. A very short explanation will suffice.

5. (7 points) Give an example of an abelian, but non-cyclic, group of order 16. Explain.

6. (7 points) Let H be the subgroup $<(1,2,3)>$ of the group $G = A_4$, and let S be the set of left cosets of H in G. Define multiplication on S by $(g_1H)(g_2H) = (g_1g_2)H$ for all g_1 and g_2 in G. Is S a group? Explain very thoroughly.

7. (9 points) Let N be a normal subgroup of the group G and let H be any subgroup of G. Let HN be the subset $\{hn \mid h \in H \text{ and } n \in N\}$ of G.
   (a) Prove that HN is a subgroup of G.
   (b) Prove that N is a normal subgroup of HN.
   (c) Let $\varphi\colon H \to \frac{HN}{N}$ be the group homomorphism which is given as the composition of inclusion $H \to HN$, followed by the natural quotient map $HN \to \frac{HN}{N}$. What is the kernel of φ?
   (d) Apply the First Isomorphism Theorem to φ.
   (You just proved the "Second Isomorphism Theorem".)

8. (7 points) Let V_4 be the subset $\{\text{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ of S_4. It is true that V_4 is a normal subgroup of S_4; however, you do not have to prove this. What familiar group is isomorphic to $\frac{S_4}{V_4}$? Explain.

9. (7 points) List the elements of the group $S_3 \times U_4$. What is the order of each element?

10. (7 points) Suppose that G is a group with at least two elements and that the only subgroups of G are $\{\text{id}\}$ and G. What is G? Say as much as you can. Prove your statement.

11. (7 points) Let G be a finite group of order n. Let g be an element of G. Prove that g^n is equal to the identity element of G.

12. (7 points) Let a and b be elements of finite order in the group G. State and prove an interesting statement which gives the order of ab in terms of the order of a and the order of b.

13. (7 points) Suppose that S and T are sets and $\phi\colon S \to T$ and $\theta\colon T \to S$ are functions with $\theta \circ \phi$ equal to the identity function on S.
   (a) Does θ have to be one-to-one? PROVE or give a COUNTEREXAMPLE.
   (b) Does ϕ have to be onto? PROVE or give a COUNTEREXAMPLE.

14. (7 points) Prove that $\frac{\mathbb{R}}{\mathbb{Z}} \cong U$, where U is the unit circle in $(\mathbb{C} \setminus \{0\}, \times)$ and $\mathbb{R}$ and $\mathbb{Z}$ are groups under addition.