

**Math 546, Exam 2 SOLUTIONS, Spring 2010**

Write everything on the blank paper provided.

**You should KEEP this piece of paper.**

If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **6** problems.

Write **coherently in complete sentences.**

**No Calculators or Cell phones.**

I will post the solutions later today.

1. **(9 points) Let  $G$  be a group. Prove that the identity element in  $G$  is unique.**

Suppose that  $e$  and  $e'$  both are identity elements in  $G$ . Consider the product  $ee'$ . The fact that  $e$  is an identity element of  $G$  tells us that  $ee' = e'$ . The fact that  $e'$  is an identity element of  $G$  tells us that  $ee' = e$ . We conclude that  $e = ee' = e'$ .

2. **(7 points) Define *centralizer*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.**

Let  $g$  be an element in a group  $G$ . The *centralizer* of  $g$  in  $G$  is the set of all elements in  $G$  which commute with  $g$ . In other words,

$$C(g) = \{x \in G \mid xg = gx\}.$$

3. **(7 points) Define *order*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.**

Let  $g$  be an element in a group  $G$ . The *order* of  $g$  is the least positive integer  $n$  for which  $g^n$  is equal to the identity element of  $G$ . If  $g^n$  is never equal to the identity element of  $G$ , for any positive integer  $n$ , then  $g$  has infinite order.

4. (9 points) Let  $G$  be an Abelian group and let  $H$  be the subset

$$H = \{g \in G \mid g^2 = \text{id}\}$$

of  $G$ . Does  $H$  have to be a subgroup of  $G$ ? If yes, then prove the claim. If no, then give an example.

YES,  $H$  is always a subgroup of  $G$  as we prove below.

**Closure.** Take  $g_1$  and  $g_2$  in  $H$ . We see that

$$(g_1g_2)(g_1g_2) = (g_1g_1)(g_2g_2) = \text{id}.$$

(The first equality is due to the fact that  $G$  is an Abelian group. The second equality is due to the fact that  $g_1$  and  $g_2$  are both in  $H$ .) Thus, the product  $g_1g_2$  is also in  $H$ .

**Inverses.** Take  $g \in H$ . It follows that  $g^2 = \text{id}$ ; thus  $g$  is the inverse of  $g$  and the inverse of  $g$  (which is  $g$  itself) is also in  $H$ .

**Identity.** The identity element squares to the identity element; thus,  $\text{id} \in H$ .

5. (9 points) Let  $G$  be a group and let  $H$  be the subset

$$H = \{g^3 \in G \mid g \in G\}$$

of  $G$ . Does  $H$  have to be a subgroup of  $G$ ? If yes, then prove the claim. If no, then give an example.

NO,  $H$  is not always a subgroup of  $G$ . Consider  $G = D_3$ . In this case,

$$H = \{\text{id}, \sigma, \sigma\rho, \sigma\rho^2\}$$

and  $H$  is not a group because  $\sigma$  and  $\sigma\rho$  are in  $H$  but the product  $\sigma(\sigma\rho) = \rho$  is not in  $H$ .

6. (9 points) List three subgroups of  $D_4$  of order four. (No details are necessary.)

$$\{\text{id}, \rho, \rho^2, \rho^3\}, \quad \{\text{id}, \sigma, \rho^2, \sigma\rho^2\}, \quad \{\text{id}, \sigma\rho, \rho^2, \sigma\rho^3\}.$$