Math 546, Exam 2, Fall, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

No calculators, cell phones, computers, notes, etc.

Make your work correct, complete, and coherent.

The exam is worth 50 points. Each problem is worth 10 points.

We use $\mathbb{Q}$ to mean the group of rational numbers under addition and $\mathcal{G}$ to mean the group of rigid motions of the plane under composition.

The solutions will be posted later today.

(1) Suppose G is a group and every proper subgroup of G is cyclic. Does G have to be cyclic? If so, prove the statement. If not, give an example.

(2) Let H and K be non-zero subgroups of the group $\mathbb{Q}$. Does the intersection of H and K have to be non-zero? If so, prove the statement. If not, give an example.

(3) Recall the group D_4, which is the subgroup of $\mathcal{G}$ generated by σ and ρ, where σ is reflection across the x-axis and ρ is rotation by $\pi/2$ radians counterclockwise fixing the origin. Recall that D_4 has eight distinct elements $\sigma^i\rho^j$ with $i \in \{0,1\}$ and $j \in \{0,1,2,3\}$ and that the generators of D_4 satisfy
$$\sigma^2 = \mathrm{id}, \quad \rho^4 = \mathrm{id}, \quad \rho\sigma = \sigma\rho^3.$$
Which elements of D_4 commute with $\sigma\rho$? Prove your answer.

(4) Let $(G, *)$ be a group and $H = \{g \in G \mid g * g = \mathrm{id}\}$. Does H have to be a subgroup of G? If so, prove the statement. If not, give an example.

(5) Let H be a subgroup of the group $(G, *)$ and let g_1 and g_2 be two elements of G. Suppose that the cosets $g_1 * H$ and $g_2 * H$ have an element in common. Prove that $g_1 * H$ is a subset of $g_2 * H$.