Math 546, Exam 1, Fall, 2022, Solutions

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

No calculators, cell phones, computers, notes, etc.

Make your work correct, complete, and coherent.

The exam is worth 50 points. Each problem is worth 10 points.

The solutions will be posted later today.

(1) **Suppose H and K are subgroups of the group G. Is the union $H \cup K$ always a subgroup of G? If so, prove the statement. If not, give an example.**

The union of H and K is not always a subgroup of G. Consider the group G of 2×2 matrices with integer entries under addition. Let H and K be the subgroups

$$H = \left\{ \begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix} \middle| n \in \mathbb{Z} \right\} \quad \text{and} \quad K = \left\{ \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} \middle| n \in \mathbb{Z} \right\}.$$

We see that $H \cup K$ is not a group because $H \cup K$ is not closed under the group operation. Indeed, if

$$h = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then $h \in H$ and $k \in K$ so h and k are both in $H \cup K$; however, $h + k = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is not in $H \cup K$.

(2) **Let $(G, *)$ be a group, K be a subgroup of G, g be a fixed element of G with inverse g^{-1}, and $H = \{g * k * g^{-1} \mid k \in K\}$. Prove that H is a group.**

The group K is non-empty; hence the set H is also non-empty. We must show that H is closed under $*$ and that if $h \in H$, then the inverse of h in G is also in H.

We first show that H is closed under $*$. Take two elements h_1 and h_2 in H. So, there exist k_1 and k_2 in K with $h_i = g * k_i * g^{-1}$. We calculate that $h_1 * h_2 = (g * k_1 * g^{-1}) * (g * k_2 * g^{-1}) = g * k_1 * k_2 * g^{-1}$, by associativity and the property of inverses. The most recent element is in H because it has the correct form. Indeed, $k_1 * k_2$ is an element of K because K is closed under $*$.

Now we show that if $h \in H$, then the inverse of h in G is also in H. The element h of H is equal to $g * k * g^{-1}$ for some $k \in K$. The inverse of h in G is $g * k^{-1} * g^{-1}$ (because the product of $g * k * g^{-1}$ and $g * k^{-1} * g^{-1}$ is id, in either order, where k^{-1} is the inverse of K). Notice that k^{-1} is in K because K is a group. We conclude that $g * k^{-1} * g^{-1}$ is in H. Thus, if $h \in H$, then the inverse of h in G is also in H.

(3) **Let σ and τ be the following elements of $\mathrm{Sym}(\{1,2,3,4\})$:**

$$\sigma(1) = 2, \quad \sigma(2) = 1, \quad \sigma(3) = 3, \quad \sigma(4) = 4,$$

$$\tau(1) = 2, \quad \tau(2) = 3, \quad \tau(3) = 4, \quad \tau(4) = 1.$$

What is the smallest positive integer n with $(\sigma \circ \tau)^n = \mathrm{id}$?

Observe that $\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, $(\sigma \circ \tau)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$, and $(\sigma \circ \tau)^3 = \mathrm{id}$. So 3 is the smallest positive integer n with $(\sigma \circ \tau)^n = \mathrm{id}$.

(4) **Let σ and τ be the following elements of $\mathrm{Sym}(\{1,2,3,4\})$:**

$$\sigma(1) = 3, \quad \sigma(2) = 2, \quad \sigma(3) = 1, \quad \sigma(4) = 4,$$

$$\tau(1) = 1, \quad \tau(2) = 4, \quad \tau(3) = 3, \quad \tau(4) = 2.$$

Let H the smallest subgroup of $\mathrm{Sym}(\{1,2,3,4\})$ which contains σ and τ. What is the multiplication table for H?

Observe that $\sigma \circ \tau$ and $\tau \circ \sigma$ both equal $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$. Observe also that $\sigma^2 = \tau^2 = (\sigma \circ \tau)^2 = \mathrm{id}$. It is now clear that H is another example of the Klein 4-group

	id	σ	τ	$\sigma \circ \tau$
id	id	σ	τ	$\sigma \circ \tau$
σ	σ	id	$\sigma \circ \tau$	τ
τ	τ	$\sigma \circ \tau$	id	σ
$\sigma \circ \tau$	$\sigma \circ \tau$	τ	σ	id

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(5) **Let $(G, *)$ be a group and $H = \{g \in G \mid g * g = \mathrm{id}\}$. Does H have to be a subgroup of G? If so, prove the statement. If not, give an example.**

NO. Let be the group S_3. In this case, H is the set

$$\left\{ \mathrm{id}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}.$$

We see that H is not a group because, H is not closed

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} \circ \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix},$$

which is not in H.