Solution to the Quiz for April 14, 2003

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

The determinant of $A - \lambda I$ is

$$\det \begin{bmatrix} 2-\lambda & 1 & 2 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2(3-\lambda).$$

The eigenvalues of A are 2 and 3. The eigenspace of A which belongs to $\lambda = 2$ is the null space of

$$A - 2I = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Replace row 2 by row 2 minus row 1 to get

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace of A which belongs to 2 is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, with

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= \quad -2x_3 \\ x_3 &= \qquad x_3. \end{aligned}$$

The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ are a basis for the eigenspace of A which belongs to 2.

Check. We see that

$$A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}2\\0\\0\end{bmatrix}=2\begin{bmatrix}1\\0\\0\end{bmatrix}.\checkmark$$

We also see that

$$A\begin{bmatrix}0\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\-4\\2\end{bmatrix}=2\begin{bmatrix}0\\-2\\1\end{bmatrix}.\checkmark$$

The eigenspace of A which belongs to $\lambda=3$ is the null space of

$$A-3I=\begin{bmatrix}-1&1&2\\0&0&2\\0&0&-1\end{bmatrix}.$$

Replace row 2 by $1/2$ row 2:

$$\begin{bmatrix}-1&1&2\\0&0&1\\0&0&-1\end{bmatrix}.$$

Replace row 1 by row 1 minus 2 row 2. Replace row 3 by row 3 plus row 2:

$$\begin{bmatrix}-1&1&0\\0&0&1\\0&0&0\end{bmatrix}.$$

Multiply row 1 by minus 1:

$$\begin{bmatrix}1&-1&0\\0&0&1\\0&0&0\end{bmatrix}.$$

The eigenspace of A which belongs to 3 is the set of all vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, with

$$\begin{aligned}x_1&=x_2\\x_2&=x_2\\x_3&=0.\end{aligned}$$

The vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for the eigenspace of A which belongs to 3.

Check. We see that

$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} . \checkmark$$