Quiz 8, March 24, 2016

Let A be an $m \times m$ non-singular matrix and B be an $m \times n$ matrix.

(a) Prove that AB and B have the same null space.

(b) Prove that the column space of AB and the column space of B have the same dimension.

Answer: (a) We first prove that the null space of B is a subset of the null space of AB. If v is in the null space of B, then $Bv = 0$; hence, $ABv = 0$. Thus, v is also in the null space of AB.

Now we show that the null space of AB is contained in the null space of B. If v is in the null space of AB then $ABv = 0$. The matrix A is non-singular and $A(Bv) = 0$. It follows that $Bv = 0$; hence v is also in the null space of B.

(b) The rank nullity theorem tells us that the dimension of the column space of AB is equal to the number of columns of AB minus the dimension of the null space of AB. The number of columns of AB is the same as the number of columns of B. We saw in (a) that AB and B have the same null space. Thus the dimension of the column space of AB is equal to the number of columns of B minus the dimension of the null space of B and the rank nullity theorem guarantees that the most recent number is the dimension of the column space of B.