Let A and B be $n \times n$ matrices with AB non-singular. Prove that A and B are both non-singular.

ANSWER: We first show that B is non-singular. Suppose that v is a vector with $Bv = 0$. Multiplication by A gives $ABv = A0 = 0$. The matrix AB is non-singular and $ABv = 0$. It follows that $v = 0$.

Now we show that A is non-singular. Suppose that v is a vector with $Av = 0$. We saw above that the matrix B is non-singular. It follows from the non-singular matrix theorem that B is invertible. Let B^{-1} be the inverse of B. We have $0 = Av = AB(B^{-1}v)$. The matrix AB is non-singular; so, $B^{-1}v = 0$. Multiply by B to see that $BB^{-1}v = B0 = 0$. Thus, v, which is equal to $BB^{-1}v$, is the zero vector.