Quiz 4, February 11, 2016

Determine the conditions on the numbers a and b which cause the vectors

$$v_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\a\\3\end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix}0\\2\\b\end{bmatrix}$$

to be linearly dependent.

Answer We solve the system of equations $Ac = 0$, where

$$A = \begin{bmatrix}1 & 1 & 0\\2 & a & 2\\1 & 3 & b\end{bmatrix} \quad c = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}, \quad \text{and} \quad 0 = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

We apply Gaussian elimination to

$$\begin{bmatrix}1 & 1 & 0\\2 & a & 2\\1 & 3 & b\end{bmatrix}.$$

(There is no need to augment the matrix with a column of zeros.) Replace $R2 \mapsto R2 - 2R1$ and $R3 \mapsto R3 - R1$ to obtain

$$\begin{bmatrix}1 & 1 & 0\\0 & a-2 & 2\\0 & 2 & b\end{bmatrix}.$$

Exchange rows 2 and 3 to obtain

$$\begin{bmatrix}1 & 1 & 0\\0 & 2 & b\\0 & a-2 & 2\end{bmatrix}.$$

Replace $R3 \mapsto R3 - \frac{a-2}{2}R2$ to obtain

$$\begin{bmatrix}1 & 1 & 0\\0 & 2 & b\\0 & 0 & 2-\frac{b(a-2)}{2}\end{bmatrix}.$$

If $2 - \frac{b(a-2)}{2} \neq 0$, then the only vector c with $Ac = 0$ is $c = 0$ and v_1, v_2, v_3 are linearly independent.

If $2 - \frac{b(a-2)}{2} = 0$, then there are many vectors c with $Ac = 0$ and v_1, v_2, v_3 are linearly dependent.