51 35

4. True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If A is a 2×2 matrix and c is a constant, then $\det(cA) = c \det A$.

False Take $A = I$ and $c = 2$

$\det cA = 4 \quad c \det A = 2$

5. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ with

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\3\end{bmatrix}.$$

Find a matrix A with $T(v) = Av$ for all $v \in \mathbb{R}^2$.

We first solve $\begin{bmatrix}x\\y\end{bmatrix} = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\2\end{bmatrix}$ i.e. $\left[\begin{array}{cc|c}1&1&x\\1&2&y\end{array}\right] \xrightarrow{R2 \mapsto R2 - R1} \left[\begin{array}{cc|c}1&1&x\\0&1&y-x\end{array}\right] \xrightarrow{R1 \mapsto R1 - R2} \left[\begin{array}{cc|c}1&0&2x-y\\0&1&y-x\end{array}\right]$

Thus $\begin{bmatrix}x\\y\end{bmatrix} = (2x-y)\begin{bmatrix}1\\1\end{bmatrix} + (y-x)\begin{bmatrix}1\\2\end{bmatrix}$

so $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = (2x-y)T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + (y-x)T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = (2x-y)\begin{bmatrix}1\\2\\3\end{bmatrix} + (y-x)\begin{bmatrix}1\\-2\\3\end{bmatrix}$

$$= \begin{bmatrix}x\\4x-2y \quad -2y+2x\\6x-3y+3y-3x\end{bmatrix} = \begin{bmatrix}x\\6x-4y\\3x\end{bmatrix} = \begin{bmatrix}1&0\\6&-4\\3&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

So $A = \begin{bmatrix}1&0\\6&-4\\3&0\end{bmatrix}$