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2. Consider the vectors

$$u_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad u_2 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \quad u_3 = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \quad \text{and} \quad u_4 = \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}.$$

(a) (2 points) Do the vectors u_1, u_2, u_3, u_4 form an orthogonal set? Why?

(b) (9 points) Express $v = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ as a linear combination of u_1, u_2, u_3, u_4.

(c) (9 points) Find the inverse of $\begin{bmatrix}1 & 1 & 0 & -1\\1 & 0 & 1 & 1\\1 & -1 & 0 & -1\\1 & 0 & -1 & 1\end{bmatrix}$.

a) yes because $u_1^T u_2 = 0$, $u_1^T u_3 = 0$, $u_1^T u_4 = 0$, $u_2^T u_3 = 0$, $u_2^T u_4 = 0$ $u_3^T u_4 = 0$

(b) If $v = c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4$, then $u_1^T v = c_1 u_1^T u_1$ i.e. $10 = 4c_1$

So $c_1 = \frac{10}{4} = \frac{5}{2}$ $\quad u_2^T v = c_2 u_2^T u_2 \quad -2 = 2c_2 \quad -1 = c_2$

$u_3^T v = c_3 u_3^T u_3 \quad -2 = c_3 2 \quad -1 = c_3 \quad u_4^T v = c_4 u_4^T u_4 \quad 2 = c_4 \cdot 4 \quad c_4 = \frac{1}{2}$

So $\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \frac{5}{2} u_1 - u_2 - u_3 + \frac{1}{2} u_4$

(c) $\begin{bmatrix}1&1&1&1\\1&0&-1&0\\0&1&0&-1\\-1&1&-1&1\end{bmatrix}\begin{bmatrix}1&1&0&-1\\1&0&1&1\\1&-1&0&-1\\1&0&-1&1\end{bmatrix} = \begin{bmatrix}4&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix}$

So $\begin{bmatrix}\frac{1}{4}&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{2}&0\\0&0&0&\frac{1}{4}\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&0&-1&0\\0&1&0&-1\\-1&1&-1&1\end{bmatrix} = \begin{bmatrix}\frac{1}{4}&\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\\\frac{1}{2}&0&-\frac{1}{2}&0\\0&\frac{1}{2}&0&-\frac{1}{2}\\-\frac{1}{4}&\frac{1}{4}&-\frac{1}{4}&\frac{1}{4}\end{bmatrix}$ is the inverse $\begin{bmatrix}1&1&0&-1\\1&0&1&1\\1&-1&0&-1\\1&0&-1&1\end{bmatrix}$