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8. True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.)
If $v_1, v_2, v_3, v_4$ are in $\mathbb{R}^4$ and $v_3$ is *not* a linear combination of $v_1, v_2, v_4$, then $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

False

$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $V_4 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$V_3$ is not a linear combination of $V_1, V_2, V_4$

But $V_1, V_2, V_3, V_4$ is linearly dependent because $2V_1 - V_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

9. Are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 3 \\ 2 \\ 8 \\ -3 \end{bmatrix}$$

linearly dependent or linearly independent? Show your work. Check your answer.

I look for $c_1, c_2, c_3, c_4$ with $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$. $\bigstar$

$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 1 & -3 \end{bmatrix}$ $R3 \mapsto R3 - R1$ $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 1 & 1 & -3 \end{bmatrix}$ $R1 \mapsto R1 + R3$, $R4 \mapsto R4 - R2$ $\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 1 & -5 \end{bmatrix}$ $R4 \mapsto R4 + R3$ $\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \mapsto -R_3$ $\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The vectors are linearly dependent because

$c_1 = -8c_4$
$c_2 = -2c_4$
$c_3 = 5c_4$
$c_4 = c_4$

gives a nontrivial solution to $\bigstar$. In particular,

$$-8v_1 - 2v_2 + 5v_3 + v_4 = \begin{bmatrix} -8+0+5+3 \\ 0-2+0+2 \\ -8+0+0+8 \\ 0-2+5-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

my check