8. Consider the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$, which is given by $T(v) = Mv$ for all $v \in \mathbb{R}^4$, where $M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$. Is T one-to-one? Is T onto? Explain your answer.

We are asked about the number of solutions of $Mx = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Look at $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 1 & 1 & 1 & 2 & b \\ 2 & 2 & 2 & 3 & c \end{array}\right]$ $\xrightarrow[R2 \mapsto R2 - R1]{R3 \mapsto R3 - 2R1}$ $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 0 & 0 & 1 & b-a \\ 0 & 0 & 0 & 1 & c-2a \end{array}\right]$ $\xrightarrow{R3 \mapsto R3 - R2}$ $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 0 & 0 & 1 & b-a \\ 0 & 0 & 0 & 0 & c-b-a \end{array}\right]$

So $Mx = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has at least one solution only if $c = a + b$ so T is not onto.

$Mx = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has more than one solution so T is not one-to-one.

9. Define "onto". Use complete sentences.

The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if for each $w \in \mathbb{R}^m$, there exists at least one $v \in \mathbb{R}^n$ with $T(v) = w$.

10. Define "linear transformation". Use complete sentences.

The function $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if $T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$ for all $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$.