MATH 544, HOMEWORK, SPRING 2022

(1) Find the general solution of the following system of linear equations:

$$\begin{array}{rrrrrl} x_1 & + \; x_2 & & & - \; x_5 & = 1 \\ & x_2 & +2x_3 & + \; x_4 & +3x_5 & = 1 \\ x_1 & & - \; x_3 & + \; x_4 & + \; x_5 & = 0. \end{array}$$

Also find **three** particular solutions of this system of equations. **Be sure to check** that all three of your particular solutions really satisfy the original system of linear equations.

(2) Find the general solution of the following system of linear equations:

(3) Find the general solution of the following system of linear equations:

$$\begin{array}{rrl} x_1 & +x_2 & = 4 \\ x_1 & +2x_2 & = 6 \\ 5x_1 & +8x_2 & = 26 \end{array}$$

(4) (a) Find all values of a for which the following system of equations has no solution.

(b) Find all values of a for which the following system of equations has exactly one solution.

(c) Find all values of a for which the following system of equations has an infinite number of solutions.

$$\begin{array}{rrl} x_1 & +2x_2 & = -3 \\ ax_1 & -2x_2 & = \;\; 5 \end{array}$$

(5) Compute

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

(6) Find scalars a_1 and a_2 so that $a_1 r + a_2 s = t$, where

$$r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad s = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad t = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

(7) Find x so that $x^{\mathrm{T}} a = 6$ and $x^{\mathrm{T}} b = 2$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad a = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

(8) True or False. If the statement is true, then prove the statement. If the statement is false, then give a counterexample. If A and B are 2×2 symmetric matrices, then AB is a symmetric matrix.

(9) True or False. If the statement is true, then prove the statement. If the statement is false, then give a counterexample. If A and B are 2×2 matrices with $A^2 = AB$, then $A = B$.

(10) Express $b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ as a linear combination of $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(11) Let v_1, v_2, and v_3 be non-zero vectors in $\mathbb{R}^4$. Suppose that $v_i^{\mathrm{T}} v_j = 0$ for all subscripts i and j with $i \neq j$. Prove that v_1, v_2, and v_3 are linearly independent.

(12) Let A and B be symmetric $n \times n$ matrices. Suppose that AB is also a symmetric matrix. Prove that $AB = BA$.

(13) Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^5$. Suppose that v_1, v_2, v_3 are linearly dependent. Do the vectors v_1, v_2, v_3, v_4 have to be linearly dependent? If yes, prove the result. If no, show an example.

(14) True or False. (If the statement is true, then prove the statement. If the statement is false, then give a counterexample.)
If v_1, v_2, v_3, v_4 are in $\mathbb{R}^4$ and v_3 is *not* a linear combination of v_1, v_2, v_4, then the vectors v_1, v_2, v_3, v_4 are linearly independent.

(15) Let v_1, v_2, and v_3 be vectors in $\mathbb{R}^n$ and M be an $n \times n$ matrix. Suppose the vectors v_1, v_2, v_3 are linearly independent. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.

(16) Let v_1, v_2, and v_3 be vectors in $\mathbb{R}^n$ and M be a nonsingular $n \times n$ matrix. Suppose the vectors v_1, v_2, v_3 are linearly independent. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.

(17) True or False. If the statement is true, then prove the statement. If the statement is false, then give a counterexample. If A and B are 2×2 nonsingular matrices, then $A + B$ is a nonsingular matrix.

(18) True or False. If the statement is true, then prove it. If the statement is false, then give a counterexample. If A and B are singular 2×2 matrices, then $A + B$ is a singular matrix.

(19) True or False. If the statement is true, then prove the statement. If the statement is false, then give a counterexample. If A and B are 2×2 nonsingular matrices, then AB is a nonsingular matrix.

(20) Find the inverse of

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(21) Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 3 \end{bmatrix}$. Find A^{-1}.

(22) Which numbers a make $A = \begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}$ non-singular? Explain.

Instructions 0.1. In each of problems (23) to (41), decide if W is a vector space. If W is a vector space, explain why. (Whenever possible exhibit W as the null space and/or column space of some matrix.) If W is not a vector space, explain why.

(23) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 = 2x_2 \right\}$.

(24) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 - x_2 = 2 \right\}$.

(25) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 = x_2 \text{ or } x_1 = -x_2 \right\}$

(26) The instructions are given in 0.1. Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 \text{ and } x_2 \text{ are rational numbers} \right\}.$$

(27) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 = 0 \right\}$.

(28) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| |x_1| + |x_2| = 0 \right\}$

(29) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1^2 + x_2 = 1 \right\}$.

(30) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 x_2 = 0 \right\}$

(31) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_3 = 2x_1 - x_2 \right\}$.

(32) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_2 = x_3 + x_1 \right\}$

(33) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 x_2 = x_3 \right\}$.

(34) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 = 2x_3 \right\}$.

(35) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1^2 = x_1 x_2 \right\}$.

(36) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_2 = 0 \right\}$.

(37) The instructions are given in 0.1. Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 = 2x_3 \text{ and } x_2 = -x_3 \right\}.$$

(38) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_3 = x_2 = 2x_1 \right\}$.

(39) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_2 = x_3 = 0 \right\}$.

(40) Let u be a fixed vector in $\mathbb{R}^3$. The instructions are given in 0.1. Let

$$W = \left\{ x \in \mathbb{R}^3 \middle| \, u^{\mathrm{T}} x = 0 \right\}.$$

(41) The instructions are given in 0.1. Let a and b be fixed vectors in $\mathbb{R}^3$. Consider

$$W = \left\{ x \in \mathbb{R}^3 \middle| \, a^{\mathrm{T}} x = 0 \quad \text{and} \quad b^{\mathrm{T}} x = 0 \right\}.$$

(42) Let $\mathbb{V}$ be a vector space; let U and V be subspaces of $\mathbb{V}$; and let

$$W = \{ w \in \mathbb{V} \mid w = u + v \text{ for some } u \in U \text{ and } v \in V \}.$$

Is W a vector space? Justify your answer completely.

(43) Let $\mathbb{V}$ be a vector space; let U and V be subspaces of $\mathbb{V}$; and let W be the intersection of U and V. In other words,

$$W = \{ w \in \mathbb{V} \mid w \in U \quad \text{and} \quad w \in V \}.$$

Is W a vector space? Justify your answer completely.

(44) Let $\mathbb{V}$ be a vector space; let U and V be subspaces of $\mathbb{V}$; and let W be the union of U and V. In other words,

$$W = \{ w \in \mathbb{V} \mid w \in U \quad \text{or} \quad w \in V \}.$$

Is W a vector space? Justify your answer completely.

(45) Let W be the set of all continuous functions $f(x)$ defined on the closed interval $[0,1]$ with the property that $\int_0^1 f(x)dx = 0$. Is W a vector space? Explain.

(46) Let W be the set of all twice differentiable functions $f(x)$ with the property that $f''(x) + f(x) = e^x$. Is W a vector space? Explain.

(47) Let W be the set of 2×2 matrices whose determinant is zero. Is W a vector space? Explain thoroughly.

(48) Let V be the set of non-singular 2×2 matrices. Is V a vector space? Explain your answer, thoroughly.

(49) Let V be the vector space of 3×3 skew symmetric matrices. Find a basis for V. Prove that your answer is correct. Recall that the matrix M is skew-symmetric if $M^{\mathrm{T}} = -M$.

(50) Let $\mathscr{P}_4$ be the vector space of polynomials of degree at most 4 and let W be the following subspace of $\mathscr{P}_4$:

$$W = \{ p(x) \in \mathscr{P}_4 \mid p(1) + p(-1) = 0 \quad \text{and} \quad p(2) + p(-2) = 0 \}.$$

Find a basis for W.

(51) The *trace* of the square matrix A is the sum of the numbers on its main diagonal. Let V be the set of all 3×3 matrices with trace 0. The set V is a vector space. You do NOT have to prove this. Give a basis for V. Prove that your proposed basis really is a basis.

(52) Let W be the set of polynomials $p(x)$ of degree at most three with $p(0)=2$. Is W a vector space? Explain thoroughly.

(53) Let W be the vector space of polynomials $p(x)$ of degree at most three with $p(2)=0$. Give a basis for W. Prove that your answer is correct.

(54) Let V be the vector space of all polynomials $p(x)$ of degree three or less which have the property that $p(2)=0$ and $p'(2)=0$. Find a basis for V. Explain thoroughly.

(55) Let V be the vector space of symmetric 3×3 matrices. Give a basis for V. Explain your answer.

(56) Let W be the vector space of 3×3 matrices, V be the subspace of W lower triangular matrices and U be the subspace of W of upper triangular matrices. Give a basis for U, a basis for V, a basis for $U\cap V$ and a basis for $U+V$. (Recall that the matrix M from W is upper triangular if $M_{i,j}=0$ when $j<i$ and M is *lower triangular* if $M_{i,j}=0$ when $i<j$ for the vector spaces of upper and *lower triangular* matrices.) (The symbols $U\cap V$ and $U+V$ are defined in Problem 62.)

(57) Let

$$A=\begin{bmatrix}1&2&3&1&1&3\\2&4&6&2&1&5\\2&4&6&1&2&5\\2&4&6&1&1&4\end{bmatrix}.$$

(a) Find a basis for the null space of A.
(b) Find a basis for the column space of A.
(c) Find a basis for the row space of A.
(d) Express each column of A in terms of your answer to (b).
(e) Express each row of A in terms of your answer to (c).

(58) Let $U\subseteq V$ be vector spaces. Is it always true that $\dim U\le \dim V$? If yes, prove your answer. If no, give an example.

(59) Suppose that $V\subseteq W$ are vector spaces and w_1,w_2,w_3 is a basis for W. Suppose further that w_1 and w_2 are in V, but w_3 is not in V. Do you have enough information to know the exact value of $\dim V$? If yes, prove it. If no, then give enough examples to show that $\dim V$ has not yet been determined.

(60) Suppose that $V\subseteq W$ are vector spaces and w_1,w_2,w_3,w_4 is a basis for W. Suppose further that w_1 and w_2 are in V, but neither w_3 nor w_4 is not in V. Do you have enough information to know the exact value of $\dim V$? If yes,

prove it. If no, then give enough examples to show that $\dim V$ has not yet been determined.

(61) Let $U \subseteq V \subseteq W$ be vector spaces. Suppose that v_1, v_2, v_3, v_4 is a basis for W. Suppose further that v_1, v_2, v_3 are in V, but v_4 is not in V. Suppose finally, that v_1 and v_2 are in U, but v_3 and v_4 are not in U. What is the dimension of U? Prove your answer.

(62) Let U and V be finite dimensional subspaces of the vector space W. Recall that $U \cap V$ and $U + V$ are the vector spaces

$$U \cap V = \{w \in W \mid w \in U \text{ and } w \in V\} \quad \text{and}$$

$$U + V = \{w \in W \mid \text{ there exists } u \in U \text{ and } v \in V \text{ with } w = u + v\}.$$

Give a formula which relates the following vector space dimensions $\dim U$, $\dim V$, $\dim(U \cap V)$ and $\dim(U + V)$. Prove your formula.

(63) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ \sin y \end{bmatrix}$. Is T a linear transformation? Explain.

(64) Let V be the vector space of all differentiable real-valued functions which are defined on all of $\mathbb{R}$. Let W be the vector space of all real-valued functions which are defined on all of $\mathbb{R}$. Let T from V to W be the function which is given by $T(f(x)) = f'(x)$. Is T a linear transformation? Explain very thoroughly.

(65) Is the function $F : \mathbb{R}^3 \to \mathbb{R}^2$, which is defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + 3x_2 - 2x_3 \end{bmatrix},$$

a linear transformation? If so, explain why. If not, give an example to show that one of the rules of linear transformation fails to hold.

(66) Is the function $F : \mathbb{R}^2 \to \mathbb{R}^2$, which is defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \end{bmatrix},$$

a linear transformation? If so, explain why. If not, give an example to show that one of the rules of linear transformation fails to hold.

(67) True or False. (If true, give a proof. If false, give a counter example.) If v_1, v_2, v_3 are linearly dependent vectors in $\mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation, then $T(v_1), T(v_2), T(v_3)$ are linearly dependent vectors in $\mathbb{R}^4$.

(68) Yes or No. Let v_1, v_2, v_3 be vectors in $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that $T(v_1)$, $T(v_2)$, $T(v_3)$ are linearly independent vectors in $\mathbb{R}^m$. Do the vectors v_1, v_2, v_3 have to be linearly independent? If yes, prove it. If no, give an example.

(69) True or False. (If the statement is true, then prove the statement. If the statement is false, then give a counterexample.) If v_1, v_2, v_3 are linearly independent vectors in the vector space V and $T : V \to W$ is a linear transformation of vector spaces, then $T(v_1), T(v_2), T(v_3)$ are linearly independent vectors in the vector space W.

(70) Suppose that $T : \mathscr{P}_2 \to \mathscr{P}_4$ is a linear transformation, where $T(1) = x^4$, $T(x+1) = x^3 - 2x$, and $T(x^2+2x+1) = x$. Find $T(x^2+5x-1)$.

(71) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\\1\end{bmatrix}.$$

Find $T\left(\begin{bmatrix}5\\3\end{bmatrix}\right)$.

(72) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}6\\7\end{bmatrix}.$$

Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$.

(73) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = \sqrt{3}x$. Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$.

(74) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which fixes the origin and rotates the xy-plane counter-clockwise by 45 degrees. Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$.

(75) Find the eigenvalues and the eigenvectors of the matrix $A = \begin{bmatrix}1 & 3\\2 & 2\end{bmatrix}$.

(76) Find a matrix B with $B^2 = A$ for $A = \begin{bmatrix}13 & 18\\-6 & -8\end{bmatrix}$. I expect you to write down the four entries of B.

(77) Find $\lim_{n\to\infty} A^n$, where $A = \begin{bmatrix}2 & \frac{3}{2}\\-1 & -\frac{1}{2}\end{bmatrix}$.

(78) Express $v = \begin{bmatrix}8\\9\\10\end{bmatrix}$ as a linear combination of $v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, and $v_3 = \begin{bmatrix}1\\1\\-2\end{bmatrix}$. (It might be helpful to notice that v_1, v_2 and v_3 are an orthogonal set.)

(79) Find an orthogonal basis for the null space of $A = \begin{bmatrix}1 & 3 & 4 & 5\end{bmatrix}$.