MATH 544, HOMEWORK, SPRING 2022

(1) Find the general solution of the following system of linear equations:

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & & & & & - & x_5 & = 1 \\ & & x_2 & +2x_3 & & + & x_4 & +3x_5 & & = 1 \\ x_1 & & & - & x_3 & + & x_4 & + & x_5 & = 0. \end{array}$$

Also find **three** particular solutions of this system of equations. **Be sure to check** that all three of your particular solutions really satisfy the original system of linear equations.

Answer: We use the notation of augmented matrices:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{array}\right].$$

Replace row 3 with row 3 minus row 1:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & -1 & -1 & 1 & 2 & -1 \end{array}\right].$$

Replace row 1 with row 1 minus row 2 and
replace row 3 with row 3 plus row 2:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & -1 & -4 & 0 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Replace row 1 with row 1 plus 2 row 3 and
replace row 2 with row 2 minus 2 row 3:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Our matrix is in reduced row echelon form. We read the answer. The general solution of the system of equations is

$$\boxed{\begin{cases} x_1 = 0 - 3x_4 - 6x_5 \\ x_2 = 1 + 3x_4 + 7x_5 \\ x_3 = 0 - 2x_4 - 5x_5 \\ x_4 = \quad x_4 \\ x_5 = \quad\quad x_5, \text{ where } x_4 \text{ and } x_5 \text{ are free to take any value.} \end{cases}}$$

We consider the particular solutions when $x_4 = x_5 = 0$, when $x_4 = 1$ and $x_5 = 0$, and when $x_4 = 0$ and $x_5 = 1$. These solutions are

$$\boxed{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-3\\4\\-2\\1\\0\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}-6\\8\\-5\\0\\1\end{bmatrix}.}$$

We check the first particular solution:

$$\begin{array}{ccccccccc} 0 & + & 1 & & & & & - & 0 & =1\checkmark \\ & & 1 & +2(0) & + & 0 & +3(0) & & & =1\checkmark \\ 0 & & & - & (0) & + & 0 & + & 0 & =0\checkmark. \end{array}$$

We check the second particular solution:

$$\begin{array}{ccccccccc} -3 & + & 4 & & & & & - & 0 & =1\checkmark \\ & & 4 & +2(-2) & + & 1 & +3(0) & & & =1\checkmark \\ -3 & & & - & (-2) & + & 1 & + & 0 & =0\checkmark. \end{array}$$

We check the third particular solution:

$$\begin{array}{ccccccccc} -6 & + & 8 & & & & & - & 1 & =1\checkmark \\ & & 8 & +2(-5) & + & 0 & +3(1) & & & =1\checkmark \\ -6 & & & - & (-5) & + & 0 & + & 1 & =0\checkmark. \end{array}$$

(2) Find the general solution of the following system of linear equations:

$$\begin{array}{lll} x_1 & +x_2 & =4 \\ x_1 & +2x_2 & =6. \end{array}$$

Answer: Replace equation 2 with equation 2 minus equation one:

$$\begin{array}{lll} x_1 & +x_2 & =4 \\ & +x_2 & =2. \end{array}$$

Replace equation 1 with equation 1 minus equation 2:

$$\begin{array}{lll} x_1 & & =2 \\ & +x_2 & =2. \end{array}$$

The general solution of the system of equations is

$$\boxed{\begin{array}{l} x_1 = 2 \\ x_2 = 2 \end{array}}$$

We check our answer:

$$\begin{array}{lll} 2 & +2 & =4\checkmark \\ 2 & +2(2) & =6\checkmark. \end{array}$$

(3) Find the general solution of the following system of linear equations:

$$\begin{array}{lll} x_1 & +x_2 & =4 \\ x_1 & +2x_2 & =6 \\ 5x_1 & +8x_2 & =26 \end{array}$$

Answer: Replace equation 2 with equation 2 minus equation 1 and replace equation 3 with equation 3 minus 5 times equation 1:

$$\begin{aligned} x_1 \quad x_2 \quad &= 4 \\ +x_2 \quad &= 2 \\ +3x_2 \quad &= 6. \end{aligned}$$

Replace equation 1 with equation 1 minus equation 2 and
replace equation 3 with equation 3 minus 3 times equation 2

$$\begin{aligned} x_1 \qquad &= 2 \\ +x_2 \quad &= 2 \\ +0 \quad &= 0. \end{aligned}$$

The general solution of the system of equations is

$$\boxed{\begin{aligned} x_1 &= 2 \\ x_2 &= 2 \end{aligned}}$$

We check our answer:

$$\begin{array}{ccl} 2 & +2 & = 4\checkmark \\ 2 & +2(2) & = 6\checkmark \\ 5(2) & +8(2) & = 26\checkmark. \end{array}$$

(4) (a) Find all values of a for which the following system of equations has no solution.
(b) Find all values of a for which the following system of equations has exactly one solution.
(c) Find all values of a for which the following system of equations has an infinite number of solutions.

$$\begin{array}{rrl} x_1 & +2x_2 & = -3 \\ ax_1 & -2x_2 & = 5 \end{array}$$

Answer: We use augmented matrices:

$$\left[\begin{array}{cc|c} 1 & 2 & -3 \\ a & -2 & 5 \end{array}\right].$$

Replace Row 2 by Row 2 minus a times Row 1:

$$\left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & -2-2a & 5+3a \end{array}\right].$$

If $-2-2a \neq 0$, then the system of equations has a unique solution. If $-2-2a = 0$, then $a = -1$ and the bottom equation is $0x_1 + 0x_2 = 2$, which has no solution.

The system of equations has a unique solution for all a except $a = -1$. If $a = -1$, then the system of equations has no solution.

(5) Compute
$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Answer: The product is
$$\begin{bmatrix} 2(1)+3(3) \\ 1(1)+4(3) \end{bmatrix} = \boxed{\begin{bmatrix} 11 \\ 13 \end{bmatrix}}$$

(6) Find scalars a_1 and a_2 so that $a_1 r + a_2 s = t$, where
$$r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad s = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad t = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Answer: Find numbers a_1 and a_2 so that
$$a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

That is, solve the system of equations
$$\begin{aligned} a_1 + 2a_2 &= 1 \\ 3a_2 &= 4 \end{aligned}$$

Divide equation 2 by 3:
$$\begin{aligned} a_1 + 2a_2 &= 1 \\ a_2 &= \tfrac{4}{3} \end{aligned}$$

Replace equation 1 minus 2 times equation 2:
$$\boxed{\begin{aligned} a_1 \qquad &= \tfrac{-5}{3} \\ a_2 &= \tfrac{4}{3} \end{aligned}}$$

Of course, this works:
$$\frac{-5}{3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{4}{3} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \checkmark.$$

(7) Find x so that $x^{\mathrm{T}} a = 6$ and $x^{\mathrm{T}} b = 2$, where
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad a = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Answer: The equation $x^{\mathrm{T}} a = 6$ is $x_1 + 2x_2 = 6$. The equation $x^{\mathrm{T}} b = 2$ is $3x_1 + 4x_2 = 2$. We solve the system of equations
$$\begin{cases} x_1 + 2x_2 = 6 \\ 3x_1 + 4x_2 = 2. \end{cases}$$

We use an augmented matrix:
$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 3 & 4 & 2 \end{array}\right].$$

Replace Row 2 with Row 2 minus 3 Row 1:
$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & -2 & -16 \end{array}\right]$$

Multiply Row 2 by $-(1/2)$:

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 1 & 8 \end{array}\right]$$

Replace Row 1 with Row 1 minus 2 Row 2:

$$\left[\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & 1 & 8 \end{array}\right]$$

So, $x_1 = -10$ and $x_2 = 8$ and $\boxed{x = \begin{bmatrix} -10 \\ 8 \end{bmatrix}}$.

We verify:

$$x^{\mathrm{T}}a = \begin{bmatrix} -10 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -10 + 16 = 6$$

and

$$x^{\mathrm{T}}a = \begin{bmatrix} -10 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = -30 + 32 = 2.\checkmark$$

(8) True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If A and B are 2×2 symmetric matrices, then AB is a symmetric matrix.

Answer: $\boxed{\text{False.}}$ Here is an example. The matrices $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ are symmetric, but the product

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 4 & 7 \end{bmatrix}$$

is not symmetric.

(9) True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If A and B are 2×2 matrices with $A^2 = AB$, then $A = B$.

Answer: $\boxed{\text{False.}}$ Here is an example. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then

$$AA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, $A^2 = AB$, but $A \neq B$.

(10) Express $b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ as a linear combination of $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Answer: We must find c_1 and c_2 with $c_1 v_1 + c_2 v_2 = b$. We apply Gaussian Elimination to $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 4 & 8 \end{array}\right]$.

Replace R_2 with $R_2 - 2R_1$ to get $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -2 & -2 \end{array}\right]$.
Replace R_2 with $(-1/2)R_2$ to get $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right]$.
Replace R_1 with $R_1 - 3R_2$ to get

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right].$$

We see that $c_1 = 2$ and $c_2 = 1$. We conclude that

$$\boxed{b = 2v_1 + v_2},$$

and of course, this is correct because

$$2v_1 + v_2 = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} = b. \checkmark$$

(11) Let v_1, v_2, and v_3 be non-zero vectors in $\mathbb{R}^4$. Suppose that $v_i^{\mathrm{T}} v_j = 0$ for all subscripts i and j with $i \neq j$. Prove that v_1, v_2, and v_3 are linearly independent.

Answer: Suppose c_1, c_2, and c_3 are numbers with

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0. \tag{0.0.1}$$

Multiply by v_1^{T} to get

$$c_1 \cdot v_1^{\mathrm{T}} v_1 + c_2 \cdot v_1^{\mathrm{T}} v_2 + c_3 \cdot v_1^{\mathrm{T}} v_3 = 0.$$

The hypothesis tells us that $v_1^{\mathrm{T}} v_2 = 0$ and $v_1^{\mathrm{T}} v_3 = 0$. So, $c_1 \cdot v_1^{\mathrm{T}} v_1 = 0$. The hypothesis also tells us that v_1 is not zero; from which it follows that $v_1^{\mathrm{T}} v_1 \neq 0$. We conclude that $c_1 = 0$. Multiply (0.0.1) by v_2^{T} to see that $c_2 \cdot v_2^{\mathrm{T}} v_2 = 0$; hence, $c_2 = 0$, since the number $v_2^{\mathrm{T}} v_2 \neq 0$. Multiply (*) by v_3^{T} to conclude that $c_3 = 0$. We have shown that each c_i MUST be zero. We conclude that v_1, v_2, and v_3 are linearly independent.

(12) Let A and B be symmetric $n \times n$ matrices. Suppose that AB is also a symmetric matrix. Prove that $AB = BA$.

Answer: When all of the listed hypotheses hold, then we have

$$AB = (AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}} = BA.$$

The first equality holds because AB is a symmetric matrix. The second equality holds for all matrices – we proved this result in class. The last equality holds because B and A both are symmetric matrices.

(13) Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^5$. Suppose that v_1, v_2, v_3 are linearly dependent. Do the vectors v_1, v_2, v_3, v_4 HAVE to be linearly dependent? If yes, PROVE the result. If no, show an EXAMPLE.

Answer: $\boxed{\text{Yes.}}$ We are told that there are numbers c_1, c_2, c_3, not all zero, with

$$c_1v_1 + c_2v_2 + c_3v_3 = 0.$$

Take the old numbers c_1, c_2, c_3 together with $c_4 = 0$. We now have

$$c_1v_1 + c_2v_2 + c_3v_3 + 0v_4 = 0,$$

and at least one of the coefficients is non-zero. The vectors v_1, v_2, v_3, v_4 are linearly dependent.

(14) True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.)
If v_1, v_2, v_3, v_4 are in $\mathbb{R}^4$ and v_3 is *not* a linear combination of v_1, v_2, v_4, then the vectors v_1, v_2, v_3, v_4 are linearly independent.

Answer: The assertion is false. Here is an example. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Observe that $v_3 \neq c_1v_1 + c_2v_2 + c_4v_4$ for any choice of c_1, c_2, c_3; however v_1, v_2, v_3, v_4 are linearly dependent.

(15) Let v_1, v_2, and v_3 be vectors in $\mathbb{R}^n$ and M be an $n \times n$ matrix. Suppose the vectors v_1, v_2, v_3 are linearly independent. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.

Answer: NO! Here is an example.

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It is clear that v_1, v_2, v_3 are linearly independent. It is also clear that

$$Mv_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad Mv_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad Mv_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly dependent because

$$1Mv_1 + 0Mv_2 + 0Mv_3 = 0$$

is a non-trivial linear combination of Mv_1, Mv_2, Mv_3 which is equal to 0.

(16) Let v_1, v_2, and v_3 be vectors in $\mathbb{R}^n$ and M be a nonsingular $n \times n$ matrix. Suppose the vectors v_1, v_2, v_3 are linearly independent. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.

Answer: The vectors Mv_1, Mv_2, Mv_3 are linearly independent.

Proof. Suppose c_1, c_2, c_3 are numbers with

$$c_1 M v_1 + c_2 M v_2 + c_3 M v_3 = 0.$$

Use the property of scalars and the fact that matrix multiplication distributes over addition to see that

$$M(c_1 v_1 + c_2 v_2 + c_3 v_3) = 0.$$

The matrix M is nonsingular; hence, the only vector w with $Mw = 0$ is $w = 0$. Thus, $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. On the other hand, the vectors v_1, v_2, v_3 are linearly independent. It follows that c_1, c_2, c_3 must all be zero. We have proven that Mv_1, Mv_2, Mv_3 are linearly independent. □

(17) True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If A and B are 2×2 nonsingular matrices, then $A + B$ is a nonsingular matrix.

Answer: Of course the statement is false. Here is an example. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is clear that A and B are both non-singular matrices, but

$$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is a singular matrix.

(18) True or False. If the statement is true, then prove it. If the statement is false, then give a counterexample. If A and B are singular 2×2 matrices, then $A + B$ is a singular matrix.

Answer: Of course the statement is false. Here is an example. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is clear that A and B are both singular matrices, but

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a non-singular matrix.

(19) True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If A and B are 2×2 nonsingular matrices, then AB is a nonsingular matrix.

Answer: The statement is true. We prove it. We show that if v is a vector in $\mathbb{R}^2$ with $(AB)v = 0$, then $v = 0$.

Suppose $(AB)v = 0$. Matrix multiplication associates. It follows that $A(Bv) = 0$. The matrix A is nonsingular, thus Bv must be zero. The matrix B is nonsingular, thus v must be zero. □

(20) Find the inverse of

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Answer: Apply Gaussian elimination to

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Exchange rows 1 and 3 to obtain:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Replace Row 3 with Row 3 minus 2 times row 1 to obtain:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right]$$

Multiply Row 3 by -1 to obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right]$$

Replace row 1 with row 1 minus row 3:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right]$$

So,

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}}$$

Check:

$$AA^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$A^{-1}A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(21) Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 3 \end{bmatrix}$. Find A^{-1}.

Answer: Apply Gaussian elimination to

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right].$$

Replace Row 3 with Row 3 minus 3 times Row 1 to obtain:

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -7 & -3 & -3 & 0 & 1 \end{array}\right].$$

Multiply Row 2 by $1/2$ to obtain:

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & -7 & -3 & -3 & 0 & 1 \end{array}\right].$$

Replace Row 1 by Row 1 minus 4 times Row 2 and
replace Row 3 by Row 3 plus 7 times Row 2 to obtain:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & -3 & 7/2 & 1 \end{array}\right].$$

Multiply Row 3 by 2 to obtain:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -6 & 7 & 2 \end{array}\right].$$

Replace Row 2 by Row 2 minus 1/2 Row 3 to obtain:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 3 & -3 & -1 \\ 0 & 0 & 1 & -6 & 7 & 2 \end{array}\right].$$

Thus,

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & -1 \\ -6 & 7 & 2 \end{bmatrix}}$$

Check: We compute

$$AA^{-1} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & -1 \\ -6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & -1 \\ -6 & 7 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(22) Which numbers a make $A = \begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}$ non-singular? Explain.

Answer: The matrix A is non-singular if the only column vector x with $Ax = 0$ is the zero column vector. We solve $Ax = 0$ and interpret our answer. We apply Gaussian Elimination to $\begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}$. (In our heads we store the augmented column which consists entirely of zeros throughout the entire calculation!) Replace Row 2 with Row 2 minus 2 times Row 1 to get $\begin{bmatrix} 1 & 2 \\ 0 & a-4 \end{bmatrix}$. This is far enough. If $a-4$ is equal to zero, then $Ax = 0$ has an infinite number of solutions. On the other hand, if $a-4$ is not equal to zero, then the present matrix shows us that x_2 must be zero and then x_1 must be zero.

The matrix A is non-singular for every choice of a, except $a = 4$.

Instructions 0.1. In each of problems (23) to (41), decide if W is a vector space. If W is a vector space, explain why. (Whenever possible exhibit W as the null space and/or column space of some matrix.) If W is not a vector space, explain why.

(23) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 = 2x_2 \right\}$.

Answer:

The set W IS a vector space. Indeed, W is the column space of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Also, W is the null space of $\begin{bmatrix} 1 & -2 \end{bmatrix}$.

(24) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 - x_2 = 2 \right\}$.

Answer: This W is not a vector space. Indeed, W is not closed under addition because

$$v_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

are in W, but $v_1 + v_2$ is not in W.

(25) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 = x_2 \text{ or } x_1 = -x_2 \right\}$

Answer: This W is not a vector space. Indeed, W is not closed under addition because

$$v_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

are in W, but $v_1 + v_2$ is not in W.

(26) The instructions are given in 0.1. Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 \text{ and } x_2 \text{ are rational numbers} \right\}.$$

Answer: This W is not a vector space (in our class). Indeed, W is not closed under scalar multiplication:

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is in W but πv is not in W.

(27) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 = 0 \right\}$.

Answer: This W is a vector space. Indeed, this W is the column space of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Also, this W is also the null space of $\begin{bmatrix} 1 & 0 \end{bmatrix}$.

(28) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| |x_1| + |x_2| = 0 \right\}$

Answer: This set W IS a vector space. Indeed, this set W consists of exactly one vector, namely $W = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. Thus, W is the null space of the identity matrix.

(29) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1^2 + x_2 = 1 \right\}$.

Answer: This W is not a vector space. Indeed, $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in W, but $v + v$ is not in W.

(30) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 x_2 = 0 \right\}$

Answer: This W is not a vector space. Indeed

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are both in W, but $v_1 + v_2$ is not in W.

(31) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_3 = 2x_1 - x_2 \right\}$.

Answer: This W is a vector space; indeed W is the column space of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

This W is also the null space of $\begin{bmatrix} 2 & -1 & -1 \end{bmatrix}$.

(32) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_2 = x_3 + x_1 \right\}$

Answer: This W is a vector space. Indeed, this W is the column space of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$. This W is also the null space of $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$.

(33) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 x_2 = x_3 \right\}$.

Answer: This W is not a vector space. Indeed,

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are both in W, but $v_1 + v_2$ is not in W.

(34) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 = 2x_3 \right\}$.

Answer: This W is a vector space. This W is the column space of $\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, this W is also the null space of $\begin{bmatrix} 1 & 0 & -2 \end{bmatrix}$.

(35) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1^2 = x_1 x_2 \right\}$.

Answer: This W is not a vector space. Indeed,

$$v_1 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

are both in W, but $v_1 + v_2$ is not in W.

(36) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_2 = 0 \right\}$.

Answer: This W is a vector space. Indeed, this W is the column space of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. This W is also the null space of $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

(37) The instructions are given in 0.1. Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 = 2x_3 \text{ and } x_2 = -x_3 \right\}.$$

Answer: This W is a vector space. Indeed, this W is the column space of $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. This W is also the null space of $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$.

(38) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_3 = x_2 = 2x_1 \right\}$.

Answer: This W is a vector space. Indeed, this W is the column space of $\begin{bmatrix}1\\2\\2\end{bmatrix}$. This W is also the null space of $\begin{bmatrix}2 & 0 & -1\\2 & -1 & 0\end{bmatrix}$.

(39) The instructions are given in 0.1. Let $W = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \middle| x_2 = x_3 = 0 \right\}$.

Answer: This W is a vector space. Indeed, this W is the column space of $\begin{bmatrix}1\\0\\0\end{bmatrix}$. This W is also the null space of $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$.

(40) Let u be a fixed vector in $\mathbb{R}^3$. The instructions are given in 0.1. Let

$$W = \left\{ x \in \mathbb{R}^3 \middle| u^{\mathrm{T}} x = 0 \right\}.$$

Answer: This W is a vector space. Indeed, this W is the null space of u^{T}.

(41) The instructions are given in 0.1. Let a and b be fixed vectors in $\mathbb{R}^3$. Consider

$$W = \left\{ x \in \mathbb{R}^3 \middle| a^{\mathrm{T}} x = 0 \quad \text{and} \quad b^{\mathrm{T}} x = 0 \right\}.$$

Answer: This W is a vector space. Indeed, this W is the null space of

$$\begin{bmatrix} a^{\mathrm{T}} \\ \hline b^{\mathrm{T}} \end{bmatrix}.$$

(42) Let $\mathbb{V}$ be a vector space; let U and V be subspaces of $\mathbb{V}$; and let

$$W = \{w \in \mathbb{V} \mid w = u + v \text{ for some } u \in U \text{ and } v \in V\}.$$

Is W a vector space? Justify your answer completely.

Answer: This W is a vector space.

The set W is closed under addition. Take w_1 and w_2 from W. Well, $w_1 = u_1 + v_1$ and $w_2 = u_2 + v_2$ for some $u_i \in U$ and $v_i \in V$.We see that

$$w_1 + w_2 = (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2);$$

furthermore, $u_1 + u_2 \in U$ because U is a vector space and $v_1 + v_2$ is in V because V is a vector space. We conclude that $w_1 + w_2$ is equal to an element of U plus an element of V; and therefore, $w_1 + w_2$ is in W.

The set W is closed under scalar multiplication. Take $w_1 = u_1 + v_1 \in W$, as above, and $r \in \mathbb{R}$. We see that $rw_1 = ru_1 + rv_1$. The vector space U is closed under scalar multiplication; so, ru_1 is in U. Also, rv_1 is in V again because V is a vector space. Once again rw_1 has the correct form; that is rw_1 is equal to an element of U plus an element of V; therefore, rw_1 is in W.

The zero vector in $\mathbb{V}$ is equal to the zero vector of U plus the zero vector of V; and therefore, the zero vector is in W.

(43) Let $\mathbb{V}$ be a vector space; let U and V be subspaces of $\mathbb{V}$; and let W be the intersection of U and V. In other words,

$$W = \{w \in \mathbb{V} \mid w \in U \quad \text{and} \quad w \in V\}.$$

Is W a vector space? Justify your answer completely.

Answer: The set W is a vector space.
The set W is closed under addition. Take w_1 and w_2 from W. Well, w_1 and w_2 are both in U and U is a vector space. Hence U is closed under addition; so $w_1 + w_2$ is also in U. Similarly, w_1 and w_2 are both in V and V is a vector space. Hence V is closed under addition; so $w_1 + w_2$ is also in V. We have shown that $w_1 + w_2$ is in W.

The set W is closed under scalar multiplication. Take $w \in W$ and $r \in \mathbb{R}$. The vector space U is closed under scalar multiplication; so, rw is in U. Also, rw is in V again because V is a vector space. Thus rw is in both U and V; hence, rw is in W.

The zero vector in $\mathbb{V}$ is in U and V because U and V are subspaces of $\mathbb{V}$. Thus this zero vector is in the intersection of U and V, which is W.

(44) Let $\mathbb{V}$ be a vector space; let U and V be subspaces of $\mathbb{V}$; and let W be the union of U and V. In other words,

$$W = \{w \in \mathbb{V} \mid w \in U \quad \text{or} \quad w \in V\}.$$

Is W a vector space? Justify your answer completely.

Answer: The set W does is not always a vector space. Let U be the null space of $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and V be the null space of $\begin{bmatrix} 0 & 1 \end{bmatrix}$. We see that $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in U and $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is V. Thus, u and v are both in W, but $u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in either U or V; hence $u + v$ is not in W.

(45) Let W be the set of all continuous functions $f(x)$ defined on the closed interval $[0,1]$ with the property that $\int\limits_0^1 f(x)dx = 0$. Is W a vector space? Explain.

Answer: Yes, this set W is a vector space. We saw in class that the set of all continuous functions defined on the closed interval $[0,1]$ is the vector space denoted $\mathscr{C}[0,1]$. The set W is a subset of W. To verify that W is a vector space, we need only check that W satisfies the three closure properties.
The set W is closed under addition:
Take f and g in W. Observe that

$$\int_0^1 (f+g)(x)dx = \int_0^1 (f(x)+g(x))dx \qquad \text{This is the meaning of adding functions.}$$

$$= \int_0^1 f(x)dx + \int_0^1 g(x))dx \quad \text{This is a property of integration.}$$
$$= 0+0=0, \quad \text{because } f \text{ and } g \text{ are in } W.$$

Thus, $f+g$ is in W.

The set W is closed under scalar multiplication:
Take $f \in W$ and $r \in \mathbb{R}$. Observe that

$$\begin{aligned} \int_0^1 (rf)(x)dx &= \int_0^1 r(f(x))dx && \text{This is the meaning of } rf \\ &= r\int_0^1 (f(x))dx && \text{This is a property of integrals.} \\ &= r(0) = 0, && \text{because } f \text{ is in } W. \end{aligned}$$

The zero function is in W because $\int_0^1 0\,dx = 0$.

(46) Let W be the set of all twice differentiable functions $f(x)$ with the property that $f''(x)+f(x)=e^x$. Is W a vector space? Explain.

Answer: Of course, W is not a vector space. The function $g(x) = \frac{1}{2}e^x$ is in W; but $2g(x)$ is not in W.

(47) Let W be the set of 2×2 matrices whose determinant is zero. Is W a vector space? Explain thoroughly.

Answer: $\boxed{\text{No}}$, W is not a vector space. The matrices $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&0\\0&1\end{bmatrix}$ are in W, but their sum, which is $\begin{bmatrix}1&0\\0&1\end{bmatrix}$, is not in W.

(48) Let V be the set of non-singular 2×2 matrices. Is V a vector space? Explain your answer, thoroughly.

Answer: NO. The set V is not closed under addition. The matrices

$$A = \begin{bmatrix}1&0\\0&1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}-1&0\\0&-1\end{bmatrix}$$

are both in V; but the sum $A+B = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ is not in V.

(49) Let V be the vector space of 3×3 skew symmetric matrices. Find a basis for V. Prove that your answer is correct. Recall that the matrix M is skew-symmetric if $M^{\mathrm{T}} = -M$.

Answer: The matrices

$$M_1 = \begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}, \quad M_2 = \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}, \quad \text{and} \quad M_3 = \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}$$

are a basis for V.

The matrices M_1, M_2, M_3 are linearly independent. Indeed, if c_1, c_2, and c_3 are numbers with

$$c_1M_1 + c_2M_2 + c_3M_3$$

equal to the zero matrix, then

$$\begin{bmatrix} 0 & c_1 & c_2 \\ -c_1 & 0 & c_3 \\ -c_2 & -c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $c_1 = c_2 = c_3 = 0$.

The matrices M_1, M_2, M_3 span V. Indeed, a typical element of V looks like

$$M = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

and $M = aM_1 + bM_2 + cM_3$.

(50) Let $\mathscr{P}_4$ be the vector space of polynomials of degree at most 4 and let W be the following subspace of $\mathscr{P}_4$:

$$W = \{p(x) \in \mathscr{P}_4 \mid p(1) + p(-1) = 0 \quad \text{and} \quad p(2) + p(-2) = 0\}.$$

Find a basis for W.

Answer: Every element of W has the form

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

where

$$\begin{cases} p(1) + p(-1) = 0 \\ p(2) + p(-2) = 0 \end{cases}$$

In other words,

$$\begin{cases} (a_0 + a_1 + a_2 + a_3 + a_4) + (a_0 - a_1 + a_2 - a_3 + a_4) = 0 \\ (a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4) + (a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4) = 0 \end{cases}$$

In other words,

$$\begin{cases} 2a_0 + 2a_2 + 2a_4 = 0 \\ 2a_0 + 8a_2 + 32a_4 = 0 \end{cases}$$

In other words,

$$\begin{cases} a_0 + a_2 + a_4 = 0 \\ a_0 + 4a_2 + 16a_4 = 0 \end{cases}$$

Subtract Eq1 from Eq2 to get:

$$\begin{cases} a_0 + a_2 + a_4 = 0 \\ 3a_2 + 15a_4 = 0 \end{cases}$$

$$\begin{cases} a_0 + a_2 + a_4 = 0 \\ a_2 + 5a_4 = 0 \end{cases}$$

Subtract equation 2 from Eq1:

$$\begin{cases} a_0 - 4a_4 = 0 \\ \quad a_2 + 5a_4 = 0 \end{cases}$$

So a_1, a_3, a_4 are free variables and the value of a_0 and a_2 is determined by the value of the free variables: $a_0 = 4a_4$ and $a_2 = -5a_4$. So every element of W has the form $a_1x + a_3x^3 + a_4(4 - 5x^2 + x^4)$. The polynomials $\boxed{x, x^3, 4-5x^2+x^4}$ span W and are linearly independent; they form a basis for W. By the way, $4 - 5x^2 + x^4$ vanishes at $1, -1, 2, -2$.

(51) The *trace* of the square matrix A is the sum of the numbers on its main diagonal. Let V be the set of all 3×3 matrices with trace 0. The set V is a vector space. You do NOT have to prove this. Give a basis for V. Prove that your proposed basis really is a basis.

Answer: The matrices

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_5 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

are are a basis for V.

The proposed basis in linearly independent. If $c_1, \dots, c_8$ are numbers with $\sum_{i=1}^{8} c_i M_i$ equal to the zero matrix, then

$$\begin{bmatrix} c_1 + c_2 & c_3 & c_4 \\ c_5 & -c_1 & c_6 \\ c_7 & c_8 & -c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

hence all eight c's are zero.

The proposed basis spans V. A typical element of V looks like

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where $a_{11} + a_{22} + a_{33} = 0$. Observe that

$$A = -a_{22}M_1 - a_{33}M_2 + a_{12}M_3 + a_{13}M_4 + a_{21}M_5 + a_{23}M_6 + a_{31}M_7 + a_{32}M_8.$$

(52) Let W be the set of polynomials $p(x)$ of degree at most three with $p(0) = 2$. Is W a vector space? Explain thoroughly.

Answer: $\boxed{\text{No}}$, W is not a vector space. The polynomial $p(x) = 2$ is in W but the polynomial $3p(x)$, which is the constant polynomial 6, is not in W.

(53) Let W be the vector space of polynomials $p(x)$ of degree at most three with $p(2) = 0$. Give a basis for W. Prove that your answer is correct.

Answer: The polynomials

$$p_1(x) = x - 2, \quad p_2(x) = (x-2)^2, \quad \text{and} \quad p_3(x) = (x-2)^3$$

are a basis for W.

The proposed basis in linearly independent. Suppose c_1, c_2, c_3 are constants and $c_1p_1(x) + c_2p_2(x) + c_3p_3(x)$ is the zero polynomial. The derivative of the zero polynomial is the zero polynomial. Take the derivative of both sides of

$$c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = \text{the zero polynomial}$$

to get

$$c_1 + 2c_2(x-2) + 3c_3(x-2)^2 = \text{the zero polynomial}. \tag{0.1.1}$$

Plug in $x = 2$ to learn that $c_1 = 0$. Take the derivative of (0.1.1) to obtain

$$2c_2 + 6c_3(x-2) = \text{the zero polynomial}. \tag{0.1.2}$$

Plug in $x = 2$ to see that $c_2 = 0$. Take the derivative of (0.1.2) to obtain

$$6c_3 = \text{the zero polynomial}.$$

Conclude that all three c's must be zero.

The proposed basis spans W. Every polynomial of degree three or less can be written in the form

$$p(x) = a_0 + a_1(x-2) + a_2(x-2)^3 + a_3(x-2)^3. \tag{0.1.3}$$

Indeed, if $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ is a normal looking polynomial of degree three or less, then

$$q(x) = b_0 + b_1((x-2)+2) + b_2((x-2)+2)^2 + b_3((x-2)+2)^3. \tag{0.1.4}$$

Expand (0.1.4) to obtain a polynomial in the form of (0.1.3). (Of course, one could also use Taylor's Theorem to write $q(x)$ in the form of (0.1.3).)

At any rate, the polynomial $p(x)$ of (0.1.3) is in W if and only if

$$0 = p(2) = a_0.$$

Thus, $p(x)$ is in W if and only if $p(x)$ is equal to a linear combination of $p_1(x)$, $p_2(x)$, and $p_3(x)$. Thus, the proposed basis spans V.

(54) Let V be the vector space of all polynomials $p(x)$ of degree three or less which have the property that $p(2) = 0$ and $p'(2) = 0$. Find a basis for V. Explain thoroughly.

Answer: The polynomials

$$p_2(x) = (x-2)^2, \quad p_3(x) = (x-2)^3$$

are a basis for V.

The proposed basis in linearly independent. We proved in Problem (53) that $p_2(x)$ and $p_3(x)$ are part of a larger linearly independent set. Thus $p_2(x)$ and $p_3(x)$ are linearly independent.

The proposed spans V. The vector space V is a subspace of the vector space W of problem (53). Let $p(x)$ be an element of V. So $p(x)$ is in W and $p'(2) = 0$. Apply (53) to write $p(x)$ in the form

$$p(x) = c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3.$$

It follows that

$$p'(x) = c_1 + 2c_2(x-2) + 3c_3(x-2)^2 \quad \text{and} \quad 0 = p'(2) = c_1.$$

Thus $p(x)$ is a linear combination of $p_2(x) = (x-2)^2$ and $p_3(x) = (x-2)^3$. We have shown that $p_2(x), p_3(x)$ span V.

(55) Let V be the vector space of symmetric 3×3 matrices. Give a basis for V. Explain your answer.

Answer: One basis for V is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is clear that each of the listed matrices is symmetric. It is clear that every 3×3 symmetric matrix is a linear combination of the six listed matrices. It is also clear that the six listed matrices are linearly independent.

(56) Let W be the vector space of 3×3 matrices, V be the subspace of W lower triangular matrices and U be the subspace of W of upper triangular matrices. Give a basis for U, a basis for V, a basis for $U \cap V$ and a basis for $U + V$. (Recall that the matrix M from W is upper triangular if $M_{i,j} = 0$ when $j < i$ and M is *lower triangular* if $M_{i,j} = 0$ when $i < j$ for the vector spaces of upper and *lower triangular* matrices.) (The symbols $U \cap V$ and $U + V$ are defined in Problem 62.)

Answer: Let

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{31} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$E_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and}$$

$$E_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Observe that

- $E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}$ is a basis for U
- $E_{11}, E_{21}, E_{31}, E_{22}, E_{32}, E_{33}$ is a basis for V
- E_{11}, E_{22}, E_{33} is a basis for $U \cap V$, and
- $E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}$ is a basis for $U + V$.

Indeed the following statements hold.

- Each matrix $E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}$ is in U.
- The matrices $E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}$ are linearly independent.
- The matrices $E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}$ span U.

- Each matrix $E_{11}, E_{21}, E_{31}, E_{22}, E_{32}, E_{33}$ is in V.
- The matrices $E_{11}, E_{21}, E_{31}, E_{22}, E_{32}, E_{33}$ are linearly independent.
- The matrices $E_{11}, E_{21}, E_{31}, E_{22}, E_{32}, E_{33}$ span V.

- Each matrix E_{11}, E_{22}, E_{33} is in $U \cap V$.
- The matrices E_{11}, E_{22}, E_{33} are linearly independent.
- The matrices E_{11}, E_{22}, E_{33} span $U \cap V$.

- Each matrix $E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}$ is in $U + V$.
- The matrices $E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}$ are linearly independent.
- The matrices $E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}$ span $U + V$.

(57) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix}.$$

(a) Find a basis for the null space of A.
(b) Find a basis for the column space of A.
(c) Find a basis for the row space of A.
(d) Express each column of A in terms of your answer to (b).
(e) Express each row of A in terms of your answer to (c).

Answer:

We apply Gaussian Elimination to the matrix A.
Replace Row 2 with Row 2 minus Row 1;
replace Row 3 with Row 3 minus 2 times Row 1; and
replace Row 4 with Row 4 minus 2 times Row 1 to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{bmatrix}.$$

Exchange rows 2 and 3 to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{bmatrix}.$$

Replace Row 1 with Row 1 plus Row 2 and
replace Row 4 with Row 4 minus Row 2
to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

Replace Row 1 with Row 1 plus Row 3 and
replace Row 4 with Row 4 minus Row 3

to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Multiply rows 2 and 3 by -1 to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The vectors

$$\boxed{w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

The vectors

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad A_{*,4} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad A_{*,5} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}}$$

are a basis for the column space of A. Notice that I am writing $A_{*,j}$ for column j of the matrix A.

The vectors

$$\boxed{\begin{array}{l} z_1 = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \\ z_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \\ z_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}}$$

are a basis for the row space of A.

We see that

$$\boxed{A_{*,2} = 2A_{*,1}, \quad A_{*,3} = 3A_{*,1}, \quad A_{*,6} = A_{*,1} + A_{*,4} + A_{*,5}.}$$

I write $A_{i,*}$ for row i of A. We see that

$$\boxed{\begin{aligned} A_{1,*} &= z_1 + z_2 + z_3, \\ A_{2,*} &= 2z_1 + 2z_2 + z_3, \\ A_{3,*} &= 2z_1 + z_2 + 2z_3, \\ A_{4,*} &= 2z_1 + z_2 + z_3. \end{aligned}}$$

(58) Let $U \subseteq V$ be vector spaces. Is it always true that $\dim U \leq \dim V$? If yes, prove your answer. If no, give an example.

Answer: YES. Every basis for U is a linearly independent set in U; hence, every basis for U is a linearly independent set in V. One of the dimension theorems says that every linearly independent subset of a vector space V may be extended to become a basis for V. Thus, $\dim U \leq \dim V$.

(59) Suppose that $V \subseteq W$ are vector spaces and w_1, w_2, w_3 is a basis for W. Suppose further that w_1 and w_2 are in V, but w_3 is not in V. Do you have enough information to know the exact value of $\dim V$? If yes, prove it. If no, then give enough examples to show that $\dim V$ has not yet been determined.

Answer: $\boxed{\text{We know that } \dim V = 2}$. Indeed, w_1 and w_2 are linearly independent vectors in V; so w_1 and w_2 is the beginning of a basis for V and $\dim V \geq 2$. The only three dimensional subspace of W is all of W. Thus, $\dim V \leq 2$, and indeed, $\dim V = 2$.

(60) Suppose that $V \subseteq W$ are vector spaces and w_1, w_2, w_3, w_4 is a basis for W. Suppose further that w_1 and w_2 are in V, but neither w_3 nor w_4 is not in V. Do you have enough information to know the exact value of $\dim V$? If yes, prove it. If no, then give enough examples to show that $\dim V$ has not yet been determined.

Answer: NO! Let $W = \mathbb{R}^4$ and

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

• In our first example we take V to be spanned by w_1 and w_2. In this case, $\dim V = 2$.

• In our second example we take V to be spanned by w_1, w_2, and $w_3 + w_4$. In this case, $\dim V = 3$ and neither w_3 nor w_4 is in V!

(61) Let $U \subseteq V \subseteq W$ be vector spaces. Suppose that v_1, v_2, v_3, v_4 is a basis for W. Suppose further that v_1, v_2, v_3 are in V, but v_4 is not in V. Suppose finally, that v_1 and v_2 are in U, but v_3 and v_4 are not in U. What is the dimension of U? Prove your answer.

Answer: The dimension of U is $\boxed{2}$. The vector space V is a proper subspace of the four dimensional vector space W (so $\dim V \leq 3$); furthermore V contains 3 linearly independent vectors; hence, $\dim V = 3$. The vector space U is a proper subspace of the three dimensional vector space V (so $\dim U \leq 2$); furthermore U contains 2 linearly independent vectors; hence, $\dim U = 2$.

(62) Let U and V be finite dimensional subspaces of the vector space W. Recall that $U \cap V$ and $U+V$ are the vector spaces

$$U \cap V = \{w \in W \mid w \in U \text{ and } w \in V\} \quad \text{and}$$

$$U+V = \{w \in W \mid \text{ there exists } u \in U \text{ and } v \in V \text{ with } w = u+v\}.$$

Give a formula which relates the following vector space dimensions $\dim U$, $\dim V$, $\dim(U \cap V)$ and $\dim(U+V)$. Prove your formula.

Answer:

$$\boxed{\dim(U+V) = \dim U + \dim V - \dim(U \cap V)}$$

Let $a = \dim U$, $b = \dim V$, and $c = \dim U \cap V$. We will exhibit a basis of $U+V$ which contains exactly $a+b-c$ vectors. Let $z_1, \dots, z_c$ be a basis for $U \cap V$. (Every basis for $U \cap V$ has c elements.) The vectors $z_1, \dots, z_c$ are linearly independent vectors in U. Every linearly independent subset of U is part of a basis for U. Furthermore, every basis for U has a elements. Thus there are elements $u_{c+1}, \dots, u_a$ in U so that

$$z_1, \dots, z_c, u_{c+1}, \dots, u_a$$

is a basis for U.

Similarly, every linearly independent subset of V is part of a basis for V. Furthermore, every basis for V has b elements. Thus there are elements $v_{c+1}, \dots, v_b$ in V so that

$$z_1, \dots, z_c, v_{c+1}, \dots, v_b$$

is a basis for V.

We finish the proof by proving that

$$z_1, \dots, z_c, u_{c+1}, \dots, u_a, v_{c+1}, \dots, v_b \tag{0.1.5}$$

is a basis for $U+V$. (Once we have shown that (0.1.5) is a basis for $U+V$, then we will have shown that $\dim U+V = c+(a-c)+(b-c) = a+b-c$, as expected.)

We show that the vectors (0.1.5) are linearly independent. Suppose

$$A_1, \dots, A_c, B_{c+1}, \dots, B_a, C_{c+1}, \dots, C_b$$

are numbers with

$$\sum_{i=1}^{c} A_i z_i + \sum_{j=c+1}^{a} B_j u_j + \sum_{k=c+1}^{b} C_k v_k = 0.$$

Observe that

$$\sum_{i=1}^{C} A_i z_i + \sum_{j=c+1}^{a} B_j u_j = -\sum_{k=c+1}^{b} C_k v_k \tag{0.1.6}$$

is an element of $U \cap V$. The vectors $z_1, \dots, z_c$ are a basis for $U \cap V$; hence there are numbers $D_1, \dots, D_c$ with

$$\sum_{i=1}^{c} D_i z_i = -\sum_{k=c+1}^{b} C_k v_k.$$

However, the vectors $z_1, \dots, z_c, v_{c+1}, \dots, v_k$ are a basis for V; thus, the vectors

$$z_1, \dots, z_c, v_{c+1}, \dots, v_k$$

are linearly independent and $D_1 = \dots, D_c = C_1 = \dots = C_b = 0$. At this point (0.1.6) reads

$$\sum_{i=1}^{c} A_i z_i + \sum_{j=c+1}^{a} B_j u_j = 0.$$

However the vectors $z_1, \dots, z_c, u_{c+1}, \dots, u_a$ are a basis for U; thus,

$$z_1, \dots, z_c, u_{c+1}, \dots, u_a$$

are linearly independent and

$$A_1 = \dots = A_c = B_{c+1} = \dots = B_a = 0.$$

We have shown that the vectors (0.1.5) are linearly independent.

Finally, we show that the vectors (0.1.5) span $U+V$. Let w be an arbitrary element of $U+V$. It follows that $w = u+v$ for some $u \in U$ and some $v \in V$. Write u in terms of the basis $z_1, \dots, z_c, u_{c+1}, \dots, u_a$ for U. Write v in terms of the basis $z_1, \dots, z_c, v_{c+1}, \dots, v_b$ for V. Observe that you have written $w = u+v$ in terms of (0.1.5).

(63) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ \sin y \end{bmatrix}$. Is T a linear transformation? Explain.

Answer: $\boxed{\text{NO!}}$ Observe that

$$T\left(\begin{bmatrix} 0 \\ \pi/2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

but

$$T\left(2\begin{bmatrix} 0 \\ \pi/2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ \pi \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq 2T\left(\begin{bmatrix} 0 \\ \pi/2 \end{bmatrix}\right)$$

(64) Let V be the vector space of all differentiable real-valued functions which are defined on all of $\mathbb{R}$. Let W be the vector space of all real-valued functions which are defined on all of $\mathbb{R}$. Let T from V to W be the function which is given by $T(f(x)) = f'(x)$. Is T a linear transformation? Explain very thoroughly.

Answer: $\boxed{\text{YES}}$ We learned in calculus that $(f+g)' = f' + g'$. We also learned in calculus that $(rf)' = rf'$.

(65) Is the function $F : \mathbb{R}^3 \to \mathbb{R}^2$, which is defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + 3x_2 - 2x_3 \end{bmatrix},$$

a linear transformation? If so, explain why. If not, give an example to show that one of the rules of linear transformation fails to hold.

Answer: $\boxed{\text{YES}}$ Observe that

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where M is the matrix

$$M = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -2 \end{bmatrix}.$$

Now apply Example 10.2.(a) from the class notes.

(66) Is the function $F : \mathbb{R}^2 \to \mathbb{R}^2$, which is defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \end{bmatrix},$$

a linear transformation? If so, explain why. If not, give an example to show that one of the rules of linear transformation fails to hold.

Answer: $\boxed{\text{NO!}}$ Observe that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

but

$$T\left(2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \neq 2T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

(67) True or False. (If true, give a proof. If false, give a counter example.) If v_1, v_2, v_3 are linearly dependent vectors in $\mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation, then $T(v_1), T(v_2), T(v_3)$ are linearly dependent vectors in $\mathbb{R}^4$.

Answer: $\boxed{\text{TRUE!}}$ If v_1, v_2, v_3 are linearly dependent vectors, then there are numbers c_1, c_2, and c_3, not all zero with

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$

Apply the linear transformation T to both sides and use the defining properties of linear transformation to see that

$$c_1 T(v_1) + c_2 T(v_2) + c_3 T(v_3) = 0.$$

At least one of the c's remains non-zero. We conclude that $T(v_1)$, $T(v_2)$, and $T(v_3)$ are linearly dependent.

(68) **Yes or No.** Let v_1, v_2, v_3 be vectors in $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that $T(v_1)$, $T(v_2)$, $T(v_3)$ are linearly independent vectors in $\mathbb{R}^m$. Do the vectors v_1, v_2, v_3 have to be linearly independent? If yes, prove it. If no, give an example.

Answer: $\boxed{\text{YES!}}$ Suppose $c_1v_1 + c_2v_2 + c_3v_3 = 0$. Apply the linear transformation T and use the fact that T is a linear transformation to see that $c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = 0$. The vectors $T(v_1)$, $T(v_2)$, $T(v_3)$ are linearly independent; hence, $c_1 = c_2 = c_3 = 0$ and v_1, v_2, v_3 are linearly independent.

(69) True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.) If v_1, v_2, v_3 are linearly independent vectors in the vector space V and $T : V \to W$ is a linear transformation of vector spaces, then $T(v_1), T(v_2), T(v_3)$ are linearly independent vectors in the vector space W.

Answer: $\boxed{\text{False}}$ Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ which is multiplication by $\begin{bmatrix}0 & 0 & 0\end{bmatrix}$. Let

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

We see that v_1, v_2, and v_3 are linearly independent in $\mathbb{R}^3$, but $T(v_1), T(v_2)$, $T(v_3)$ are linearly dependent in $\mathbb{R}$.

(70) Suppose that $T : \mathscr{P}_2 \to \mathscr{P}_4$ is a linear transformation, where $T(1) = x^4$, $T(x+1) = x^3 - 2x$, and $T(x^2+2x+1) = x$. Find $T(x^2+5x-1)$.

Answer: Observe that $x^2+5x-1 = (x^2+2x+1)+3(x+1)-5(1)$; so

$$T(x^2+5x-1) = T(x^2+2x+1)+3T(x+1)-5T(1)$$

$$= x+3(x^3-2x)-5x^4 = \boxed{-5x^4+3x^3-5x}.$$

(71) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\\1\end{bmatrix}.$$

Find $T\left(\begin{bmatrix}5\\3\end{bmatrix}\right)$.

Answer: I see that $\begin{bmatrix}5\\3\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\end{bmatrix}$. It follows that

$$T\left(\begin{bmatrix}5\\3\end{bmatrix}\right) = T\left(3\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\end{bmatrix}\right) = 3T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$= 3\begin{bmatrix}2\\3\\1\end{bmatrix} + 2\begin{bmatrix}1\\2\\3\end{bmatrix} = \boxed{\begin{bmatrix}8\\13\\9\end{bmatrix}}.$$

(72) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}6\\7\end{bmatrix}.$$

Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$.

Answer: We see that $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right)$ and $\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix}\right)$. The function T is a linear transformation; hence,

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{2}\left[T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}6\\7\end{bmatrix}\right) = \begin{bmatrix}5\\6\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2}\left[T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix}4\\5\end{bmatrix} - \begin{bmatrix}6\\7\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\end{bmatrix}$$

and

$$\boxed{M = \begin{bmatrix}5 & -1\\6 & -1\end{bmatrix}.}$$

(73) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = \sqrt{3}x$. Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$.

Answer: The line $y = \sqrt{3}x$ makes the angle $\theta = \frac{\pi}{3}$ with the x-axis. (If need be draw the right triangle with base 1 and height $\sqrt{3}$. The hypotenuse is 2. So the angle of inclination, θ, has $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$ and $\sin\theta = \frac{\text{op}}{\text{hyp}} = \frac{\sqrt{3}}{2}$. Thus $\theta = \frac{\pi}{3}$.) It follows that

$$M = \begin{bmatrix}\cos\frac{2\pi}{3} & \sin\frac{2\pi}{3}\\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3}\end{bmatrix} = \boxed{\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}}.$$

The best check is to make sure that $Mv = v$ for some vector on $y = \sqrt{3}x$ (like for example $v = \begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}$); and $Mw = -w$ for some vector perpendicular to $y = \sqrt{3}x$ (like for example $w = \begin{bmatrix}\sqrt{3}\\ -1\end{bmatrix}$). This happens.

(74) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which fixes the origin and rotates the xy-plane counter-clockwise by 45 degrees. Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$.

Answer:

$$\boxed{M = \begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}.}$$

(75) Find the eigenvalues and the eigenvectors of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.

Answer: We compute

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 6$$
$$= \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4).$$

The eigenvalues of A are $\lambda = -1, 4$. The eigenvectors associated to $\lambda = -1$ are in the null space of $A + I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$. Elementary row operations give $\begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$. The eigenvectors of A which belong to $\lambda = -1$ are the multiples of $v = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Check that $Av = -v$. The eigenvectors associated to $\lambda = 4$ are in the null space of $A - 4I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}$. Elementary row operations give $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The eigenvectors of A which belong to $\lambda = 4$ are the multiples of $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Check that $Aw = 4w$.

(76) Find a matrix B with $B^2 = A$ for $A = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix}$. I expect you to write down the four entries of B.

Answer: The eigenvalues of A are the solutions of

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 13-\lambda & 18 \\ -6 & -8-\lambda \end{bmatrix}$$
$$= (13-\lambda)(-8-\lambda) - (18)(-6) = \lambda^2 - 5\lambda - 104 + 108 = \lambda^2 - 5\lambda + 4$$
$$= (\lambda-4)(\lambda-1).$$

The eigenvalues of A are $\lambda = 4$ and $\lambda = 1$. The eigenspace which belongs to $\lambda = 1$ is the null space of

$$A - I = \begin{bmatrix} 12 & 18 \\ -6 & -9 \end{bmatrix}.$$

Divide row 1 by 12 to get $\begin{bmatrix} 1 & 3/2 \\ -6 & -9 \end{bmatrix}$. Add 6 copies of row 1 to row 2 to get $\begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$. The eigenspace of A which belongs to 1 is the set of all vectors x with $x_1 = -\frac{3}{2}x_2$, and x_2 is arbitrary. The vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ belongs to $\lambda = 1$. We check this statement:

$$A\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -39+36 \\ +18-16 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \checkmark$$

The eigenspace which belongs to $\lambda = 4$ is the null space of

$$A - 4I = \begin{bmatrix} 9 & 18 \\ -6 & -12 \end{bmatrix}.$$

Divide row 1 by 9 to get $\begin{bmatrix} 1 & 2 \\ -6 & -12 \end{bmatrix}$. Add six copies of row 1 to row two to get $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ belongs to $\lambda = 4$. We check this statement:

$$A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -26+18 \\ 12-8 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$= 4\begin{bmatrix} -2 \\ 1 \end{bmatrix} \checkmark.$$

We now see that

$$A\begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ and $S = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$. We see that $S^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$, and that $A = SDS^{-1}$. Our answer is

$$B = S\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}S^{-1} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}.}$$

Check:

$$B^2 = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix} = A.\checkmark$$

(77) Find $\lim\limits_{n\to\infty} A^n$, where $A = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix}$.

Answer: This problem would be easy if A were a diagonal matrix. Lets diagonalize A. The eigenvalues of A satisfy

$$0 = \det(A - \lambda I) = (2-\lambda)(-\frac{1}{2} - \lambda) + \frac{3}{2} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$$

$$= (\lambda - 1)(\lambda - \frac{1}{2}).$$

The eigenvalues of A are $\lambda = 1$ and $\lambda = \frac{1}{2}$. The eigenvectors which belong to $\lambda = 1$ are the null space of $A - I = \begin{bmatrix} 1 & \frac{3}{2} \\ -1 & -\frac{3}{2} \end{bmatrix}$. Replace row 2 by row 2 plus row 1 to get $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$. The eigenspace which belongs to $\lambda = 1$ is $x_1 = -\frac{3}{2}x_2$

and x_2 can be anything. The vector $v_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is a basis for the eigenspace which belongs to $\lambda = 1$. By the way

$$Av_1 = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -6+3 \\ 3-1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = v_1,$$

as expected. The eigenspace which belongs to $\lambda = \frac{1}{2}$ is the null space of $A - \frac{1}{2} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -1 & -1 \end{bmatrix}$. Exchange the two rows: $\begin{bmatrix} -1 & -1 \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$. Replace row 2 by row 2 plus $\frac{3}{2}$ row 1: $\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$. Multiply row 1 by -1: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The eigenspace which belongs to $\lambda = \frac{1}{2}$ is $x_1 = -x_2$ and x_2 can be anything. The vector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace which belongs to $\lambda = \frac{1}{2}$. By the way

$$Av_2 = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+\frac{3}{2} \\ 1-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$= \frac{1}{2}v_2$, as expected. Now we know that

$$A \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Let

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}.$$

We calculate that $S^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$. We saw that $AS = SD$. It follows that $A = SDS^{-1}$ and

$$\lim_{n\to\infty} A^n = S \lim_{n\to\infty} D^n S^{-1} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 3 \\ -2 & -2 \end{bmatrix}}$$

(78) Express $v = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$ as a linear combination of $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. (It might be helpful to notice that v_1, v_2 and v_3 are an orthogonal set.)

Answer: Multiply each side of $v = c_1v_1 + c_2v_2 + c_3v_3$ by v_1^{T} to see that $27v = 3c_1$, so $c_1 = 9$. Similar calculations give $-1 = 2c_2$; so, $c_2 = -\frac{1}{2}$; and

$-3 = 6c_3$; so, $c_3 = -\frac{1}{2}$. We check that

$$\boxed{9v_1 - \tfrac{1}{2}v_2 - \tfrac{1}{2}v_3 = 9\begin{bmatrix}1\\1\\1\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}8\\9\\10\end{bmatrix} = v.}$$

(79) Find an orthogonal basis for the null space of $A = \begin{bmatrix}1 & 3 & 4 & 5\end{bmatrix}$.

Answer: One basis for the null space of A is

$$v_1 = \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}-4\\0\\1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}-5\\0\\0\\1\end{bmatrix}$$

Let

$$u_1 = v_1 = \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}.$$

Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}}v_2}{u_1^{\mathrm{T}}u_1}u_1 = \begin{bmatrix}-4\\0\\1\\0\end{bmatrix} - \frac{12}{10}\begin{bmatrix}-3\\1\\0\\0\end{bmatrix} = \frac{1}{5}\left(\begin{bmatrix}-20\\0\\5\\0\end{bmatrix} - 6\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix}-2\\-6\\5\\0\end{bmatrix}.$$

Let

$$u_2 = 5u_2' = \begin{bmatrix}-2\\-6\\5\\0\end{bmatrix}.$$

We check that $u_1^{\mathrm{T}}u_2 = 0$ and $Au_2 = 0$. Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}}v_3}{u_1^{\mathrm{T}}u_1}u_1 - \frac{u_2^{\mathrm{T}}v_3}{u_2^{\mathrm{T}}u_2}u_2 = \begin{bmatrix}-5\\0\\0\\1\end{bmatrix} - \frac{15}{10}\begin{bmatrix}-3\\1\\0\\0\end{bmatrix} - \frac{10}{65}\begin{bmatrix}-2\\-6\\5\\0\end{bmatrix}$$

$$= \begin{bmatrix}-5\\0\\0\\1\end{bmatrix} - \frac{3}{2}\begin{bmatrix}-3\\1\\0\\0\end{bmatrix} - \frac{2}{13}\begin{bmatrix}-2\\-6\\5\\0\end{bmatrix} = \frac{1}{26}\left(\begin{bmatrix}-130\\0\\0\\26\end{bmatrix} - 39\begin{bmatrix}-3\\1\\0\\0\end{bmatrix} - 4\begin{bmatrix}-2\\-6\\5\\0\end{bmatrix}\right)$$

$$= \frac{1}{26}\begin{bmatrix}-5\\-15\\-20\\26\end{bmatrix}.$$

Let

$$u_3 = 26u_3' = \begin{bmatrix} -5 \\ -15 \\ -20 \\ 26 \end{bmatrix}.$$

Check that $Au_3 = 0$, $u_1^{\mathrm{T}}u_3 = 0$, and $u_2^{\mathrm{T}}u_3 = 0$. Our answer is

$$\boxed{u_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -2 \\ -6 \\ 5 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -5 \\ -15 \\ -20 \\ 26 \end{bmatrix}.}$$