The variable x_2 has now been eliminated from the first and third equations. Next, we eliminate x_3 from the first and second equations and leave x_3, with coefficient 1, in the third equation:

System:	*Augmented Matrix:*
$(-1/3)E_3$: $\begin{aligned} x_1 \qquad - 5x_3 &= -3 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$	$(-1/3)R_3$: $\begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$
$E_1 + 5E_3$: $\begin{aligned} x_1 \qquad &= 7 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$	$R_1 + 5R_3$: $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$
$E_2 - 2E_3$: $\begin{aligned} x_1 \qquad &= 7 \\ x_2 \qquad &= -2 \\ x_3 &= 2 \end{aligned}$	$R_2 - 2R_3$: $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

The last system above clearly has a unique solution given by $x_1 = 7$, $x_2 = -2$, and $x_3 = 2$. Because the final system is equivalent to the original given system, both systems have the same solution. ■

The reduction process used in the preceding example is known as ***Gauss-Jordan elimination*** and will be explained in Section 1.2. Note the advantage of the shorthand notation provided by matrices. Because we do not need to list the variables, the sequence of steps in the right-hand column is easier to perform and record.

Example 7 illustrates that row equivalent augmented matrices represent equivalent systems of equations. The following corollary to Theorem 1 states this in mathematical terms.

COROLLARY Suppose $[A \mid \mathbf{b}]$ and $[C \mid \mathbf{d}]$ are augmented matrices, each representing a different $(m \times n)$ system of linear equations. If $[A \mid \mathbf{b}]$ and $[C \mid \mathbf{d}]$ are row equivalent matrices, then the two systems are also equivalent. ■

1.1 EXERCISES

Which of the equations in Exercises 1–6 are linear?

1. $x_1 + 2x_3 = 3$
2. $x_1x_2 + x_2 = 1$
3. $x_1 - x_2 = \sin^2 x_1 + \cos^2 x_1$
4. $x_1 - x_2 = \sin^2 x_1 + \cos^2 x_2$
5. $|x_1| - |x_2| = 0$
6. $\pi x_1 + \sqrt{7}x_2 = \sqrt{3}$

In Exercises 7–10, coefficients are given for a system of the form (2). Display the system and verify that the given values constitute a solution.

7. $a_{11} = 1,\quad a_{12} = 3,\quad a_{21} = 4,\quad a_{22} = -1,$
$b_1 = 7,\quad b_2 = 2;\quad x_1 = 1,\quad x_2 = 2$

8. $a_{11} = 6,\quad a_{12} = -1,\quad a_{13} = 1,\quad a_{21} = 1,$
$a_{22} = 2,\quad a_{23} = 4,\quad b_1 = 14,\quad b_2 = 4;$
$x_1 = 2,\quad x_2 = -1,\quad x_3 = 1$

9. $a_{11} = 1,\quad a_{12} = 1,\quad a_{21} = 3,\quad a_{22} = 4,$
$a_{31} = -1,\quad a_{32} = 2,\quad b_1 = 0,\quad b_2 = -1,$
$b_3 = -3;\quad x_1 = 1,\quad x_2 = -1$

10. $a_{11} = 0,\quad a_{12} = 3,\quad a_{21} = 4,\quad a_{22} = 0,$
$b_1 = 9,\quad b_2 = 8;\quad x_1 = 2,\quad x_2 = 3$

In Exercises 11–14, sketch a graph for each equation to determine whether the system has a unique solution, no solution, or infinitely many solutions.

11. $\begin{aligned} 2x + y &= 5 \\ x - y &= 1 \end{aligned}$

12. $\begin{aligned} 2x - y &= -1 \\ 2x - y &= 2 \end{aligned}$

13. $\begin{aligned} 3x + 2y &= 6 \\ -6x - 4y &= -12 \end{aligned}$

14. $\begin{aligned} 2x + y &= 5 \\ x - y &= 1 \\ x + 3y &= 9 \end{aligned}$

15. The (2×3) system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$

is represented geometrically by two planes. How are the planes related when:

a) The system has no solution?

b) The system has infinitely many solutions?

Is it possible for the system to have a unique solution? Explain.

In Exercises 16–18, determine whether the given (2×3) system of linear equations represents coincident planes (that is, the same plane), two parallel planes, or two planes whose intersection is a line. In the latter case, give the parametric equations for the line; that is, give equations of the form $x = at + b$, $y = ct + d$, $z = et + f$.

16. $\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ -2x_1 + x_2 - x_3 &= 1 \end{aligned}$

17. $\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$

18. $\begin{aligned} x_1 + 3x_2 - 2x_3 &= -1 \\ 2x_1 + 6x_2 - 4x_3 &= -2 \end{aligned}$

19. Display the (2×3) matrix $A = (a_{ij})$, where $a_{11} = 2$, $a_{12} = 1$, $a_{13} = 6$, $a_{21} = 4$, $a_{22} = 3$, and $a_{23} = 8$.

20. Display the (2×4) matrix $C = (c_{ij})$, where $c_{23} = 4$, $c_{12} = 2$, $c_{21} = 2$, $c_{14} = 1$, $c_{22} = 2$, $c_{24} = 3$, $c_{11} = 1$, and $c_{13} = 7$.

21. Display the (3×3) matrix $Q = (q_{ij})$, where $q_{23} = 1$, $q_{32} = 2$, $q_{11} = 1$, $q_{13} = -3$, $q_{22} = 1$, $q_{33} = 1$, $q_{21} = 2$, $q_{12} = 4$, and $q_{31} = 3$.

22. Suppose the matrix C in Exercise 20 is the augmented matrix for a system of linear equations. Display the system.

23. Repeat Exercise 22 for the matrices in Exercises 19 and 21.

In Exercises 24–29, display the coefficient matrix A and the augmented matrix B for the given system.

24. $\begin{aligned} x_1 - x_2 &= -1 \\ x_1 + x_2 &= 3 \end{aligned}$

25. $\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ 2x_1 \qquad - x_3 &= 1 \end{aligned}$

26. $\begin{aligned} x_1 + 3x_2 - x_3 &= 1 \\ 2x_1 + 5x_2 + x_3 &= 5 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$

27. $\begin{aligned} x_1 + x_2 + 2x_3 &= 6 \\ 3x_1 + 4x_2 - x_3 &= 5 \\ -x_1 + x_2 + x_3 &= 2 \end{aligned}$

28. $\begin{aligned} x_1 + x_2 - 3x_3 &= -1 \\ x_1 + 2x_2 - 5x_3 &= -2 \\ -x_1 - 3x_2 + 7x_3 &= 3 \end{aligned}$

29. $\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 + x_3 &= 2 \\ x_1 - x_2 + 3x_3 &= 2 \end{aligned}$

In Exercises 30–36, display the augmented matrix for the given system. Use elementary operations on equations to obtain an equivalent system of equations in which x_1 appears in the first equation with coefficient one and has been eliminated from the remaining equations. Simultaneously, perform the corresponding elementary row operations on the augmented matrix.

30. $\begin{aligned} 2x_1 + 3x_2 &= 6 \\ 4x_1 - x_2 &= 7 \end{aligned}$

31. $\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \\ -2x_1 + x_2 \qquad &= 4 \end{aligned}$

32. $\begin{aligned} x_2 + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= 6 \end{aligned}$

33. $\begin{aligned} x_1 + x_2 &= 9 \\ x_1 - x_2 &= 7 \\ 3x_1 + x_2 &= 6 \end{aligned}$

34. $\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 1 \\ -x_1 + x_2 - x_3 + x_4 &= 3 \\ -2x_1 + x_2 + x_3 - x_4 &= 2 \end{aligned}$

35. $\begin{aligned} x_2 + x_3 - x_4 &= 3 \\ x_1 + 2x_2 - x_3 + x_4 &= 1 \\ -x_1 + x_2 + 7x_3 - x_4 &= 0 \end{aligned}$

36. $\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \\ 3x_1 + x_2 &= 0 \end{aligned}$

37. Consider the equation $2x_1 - 3x_2 + x_3 - x_4 = 3$.

a) In the six different possible combinations, set any two of the variables equal to 1 and graph the equation in terms of the other two.

b) What type of graph do you always get when you set two of the variables equal to two fixed constants?

c) What is one possible reason the equation in formula (1) is called *linear*?

ADDING INTEGERS Mathematical folklore has it that Gauss discovered the formula $1+2+3+\cdots+n = n(n+1)/2$ when he was only ten years old. To occupy time, his teacher asked the students to add the integers from 1 to 100. Gauss immediately wrote an answer and turned his slate over. To his teacher's amazement, Gauss had the only correct answer in the class. Young Gauss had recognized that the numbers could be put in 50 sets of pairs such that the sum of each pair was 101:

$$(50+51)+(49+52)+(48+53)+\cdots+(1+100) = 50(101) = 5050.$$

Soon his brilliance was brought to the attention of the Duke of Brunswick, who thereafter sponsored the education of Gauss.

1.2 EXERCISES

Consider the matrices in Exercises 1–10.

a) Either state that the matrix is in echelon form or use elementary row operations to transform it to echelon form.

b) If the matrix is in echelon form, transform it to reduced echelon form.

1. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 0 & 2 & 3 \\ 2 & 0 & 1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & -2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} -1 & 4 & -3 & 4 & 6 \\ 0 & 2 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

In Exercises 11–21, each of the given matrices represents the augmented matrix for a system of linear equations. In each exercise, display the solution set or state that the system is inconsistent.

11. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{bmatrix}$

In Exercises 22–35, solve the system by transforming the augmented matrix to reduced echelon form.

22. $\begin{aligned} 2x_1 - 3x_2 &= 5 \\ -4x_1 + 6x_2 &= -10 \end{aligned}$

23. $\begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 - 4x_2 &= 1 \end{aligned}$

24. $$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 + x_2 - 4x_3 &= -3 \end{aligned}$$

25. $$\begin{aligned} x_1 + x_2 &= 2 \\ 3x_1 + 3x_2 &= 6 \end{aligned}$$

26. $$\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 - 2x_2 + 3x_3 &= 2 \end{aligned}$$

27. $$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ -3x_1 - 3x_2 + 3x_3 &= -6 \end{aligned}$$

28. $$\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= 3 \\ x_1 - 2x_2 - 2x_3 &= -2 \\ -x_1 + 16x_2 + 2x_3 &= 16 \end{aligned}$$

29. $$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 - x_2 + 7x_3 &= 8 \\ -x_1 + x_2 - 5x_3 &= -5 \end{aligned}$$

30. $$\begin{aligned} x_1 + x_2 \qquad\qquad - x_5 &= 1 \\ x_2 + 2x_3 + x_4 + 3x_5 &= 1 \\ x_1 \qquad - x_3 + x_4 + x_5 &= 0 \end{aligned}$$

31. $$\begin{aligned} x_1 \qquad + x_3 + x_4 - 2x_5 &= 1 \\ 2x_1 + x_2 + 3x_3 - x_4 + x_5 &= 0 \\ 3x_1 - x_2 + 4x_3 + x_4 + x_5 &= 1 \end{aligned}$$

32. $$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ 2x_1 + x_2 &= 3 \end{aligned}$$

33. $$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ 2x_1 + x_2 &= 2 \end{aligned}$$

34. $$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 2 \\ -x_1 - 2x_2 &= -1 \end{aligned}$$

35. $$\begin{aligned} x_1 - x_2 - x_3 &= 1 \\ x_1 \qquad + x_3 &= 2 \\ x_2 + 2x_3 &= 3 \end{aligned}$$

In Exercises 36–40, find all values a for which the system has no solution.

36. $$\begin{aligned} x_1 + 2x_2 &= -3 \\ ax_1 - 2x_2 &= 5 \end{aligned}$$

37. $$\begin{aligned} x_1 + 3x_2 &= 4 \\ 2x_1 + 6x_2 &= a \end{aligned}$$

38. $$\begin{aligned} 2x_1 + 4x_2 &= a \\ 3x_1 + 6x_2 &= 5 \end{aligned}$$

39. $$\begin{aligned} 3x_1 + ax_2 &= 3 \\ ax_1 + 3x_2 &= 5 \end{aligned}$$

40. $$\begin{aligned} x_1 + ax_2 &= 6 \\ ax_1 + 2ax_2 &= 4 \end{aligned}$$

In Exercises 41 and 42, find all values α and β where $0 \le \alpha \le 2\pi$ and $0 \le \beta \le 2\pi$.

41. $$\begin{aligned} 2\cos\alpha + 4\sin\beta &= 3 \\ 3\cos\alpha - 5\sin\beta &= -1 \end{aligned}$$

42. $$\begin{aligned} 2\cos^2\alpha - \sin^2\beta &= 1 \\ 12\cos^2\alpha + 8\sin^2\beta &= 13 \end{aligned}$$

43. Describe the solution set of the following system in terms of x_3:
$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + 2x_2 \qquad &= 5. \end{aligned}$$
For x_1, x_2, x_3 in the solution set:

a) Find the maximum value of x_3 such that $x_1 \ge 0$ and $x_2 \ge 0$.

b) Find the maximum value of $y = 2x_1 - 4x_2 + x_3$ subject to $x_1 \ge 0$ and $x_2 \ge 0$.

c) Find the minimum value of $y = (x_1 - 1)^2 + (x_2 + 3)^2 + (x_3 + 1)^2$ with no restriction on x_1 or x_2. [*Hint:* Regard y as a function of x_3 and set the derivative equal to 0; then apply the second-derivative test to verify that you have found a minimum.]

44. Let A and I be as follows:
$$A = \begin{bmatrix} 1 & d \\ c & b \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
Prove that if $b - cd \neq 0$, then A is row equivalent to I.

45. As in Fig. 1.4, display all the possible configurations for a (2×3) matrix that is in echelon form. [*Hint:* There are seven such configurations. Consider the various positions that can be occupied by one, two, or none of the symbols.]

46. Repeat Exercise 45 for a (3×2) matrix, for a (3×3) matrix, and for a (3×4) matrix.

47. Consider the matrices B and C:
$$B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$
By Exercise 44, B and C are both row equivalent to matrix I in Exercise 44. Determine elementary row operations that demonstrate that B is row equivalent to C.

48. Repeat Exercise 47 for the matrices
$$B = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

49. A certain three-digit number N equals fifteen times the sum of its digits. If its digits are reversed, the resulting number exceeds N by 396. The one's digit is one larger than the sum of the other two. Give a linear system of three equations whose three unknowns are the digits of N. Solve the system and find N.

50. Find the equation of the parabola, $y = ax^2 + bx + c$, that passes through the points $(-1, 6)$, $(1, 4)$, and $(2, 9)$. [*Hint:* For each point, give a linear equation in a, b, and c.]

51. Three people play a game in which there are always two winners and one loser. They have the

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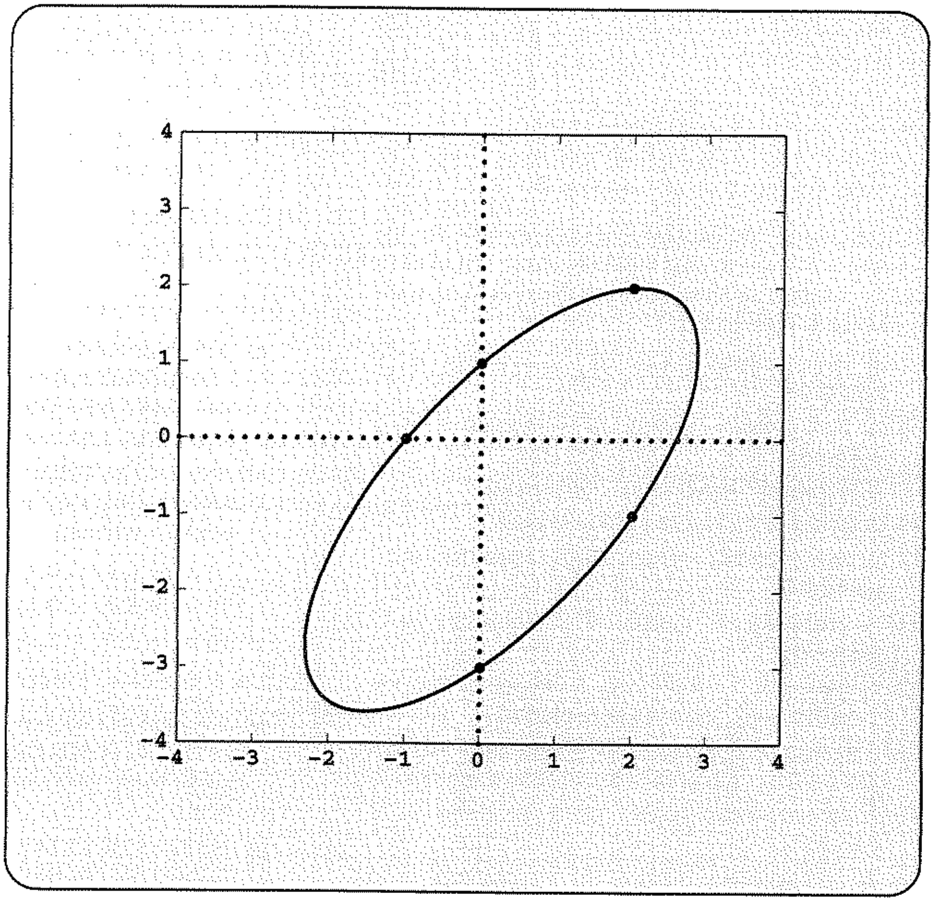

Figure 1.7 The ellipse determined by five data points, see Example 9.

1.3 EXERCISES

In Exercises 1–4, transform the augmented matrix for the given system to reduced echelon form and, in the notation of Theorem 3, determine n, r, and the number, $n - r$, of independent variables. If $n - r > 0$, then identify $n - r$ independent variables.

1.
$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= 1 \\ -2x_1 - 2x_2 + 4x_3 &= 1 \\ 2x_1 + 2x_2 + 5x_3 &= 5 \\ -2x_1 - 2x_2 - 2x_3 &= -3 \end{aligned}$$

2.
$$\begin{aligned} 2x_1 + 2x_2 &= 1 \\ 4x_1 + 5x_2 &= 4 \\ 4x_1 + 2x_2 &= -2 \end{aligned}$$

3.
$$\begin{aligned} -x_2 + x_3 + x_4 &= 2 \\ x_1 + 2x_2 + 2x_3 - x_4 &= 3 \\ x_1 + 3x_2 + x_3 &= 2 \end{aligned}$$

4.
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= 1 \\ x_1 + 2x_2 + 3x_3 + 5x_4 &= 2 \\ 2x_1 + 4x_2 + 6x_3 + x_4 &= 1 \\ -x_1 - 2x_2 - 3x_3 + 7x_4 &= 2 \end{aligned}$$

In Exercises 5 and 6, assume that the given system is consistent. For each system determine, in the notation of Theorem 3, all possibilities for the number, r of nonzero rows and the number, $n - r$, of unconstrained variables. Can the system have a unique solution?

5. $ax_1 + bx_2 = c$
$dx_1 + ex_2 = f$
$gx_1 + hx_2 = i$

6. $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3$

In Exercises 7–18, determine all possibilities for the solution set (from among infinitely many solutions, a unique solution, or no solution) of the system of linear equations described.

7. A homogeneous system of 3 equations in 4 unknowns.

8. A homogeneous system of 4 equations in 5 unknowns.

9. A system of 3 equations in 2 unknowns.

10. A system of 4 equations in 3 unknowns.

11. A homogeneous system of 3 equations in 2 unknowns.

12. A homogeneous system of 4 equations in 3 unknowns.

13. A system of 2 equations in 3 unknowns that has $x_1 = 1, x_2 = 2, x_3 = -1$ as a solution.

14. A system of 3 equations in 4 unknowns that has $x_1 = -1, x_2 = 0, x_3 = 2, x_4 = -3$ as a solution.

15. A homogeneous system of 2 equations in 2 unknowns.

16. A homogeneous system of 3 equations in 3 unknowns.

17. A homogeneous system of 2 equations in 2 unknowns that has solution $x_1 = 1, x_2 = -1$.

18. A homogeneous system of 3 equations in 3 unknowns that has solution $x_1 = 1, x_2 = 3, x_3 = -1$.

In Exercises 19–22, determine by inspection whether the given system has nontrivial solutions or only the trivial solution.

19. $2x_1 + 3x_2 - x_3 = 0$
$x_1 - x_2 + 2x_3 = 0$

20. $x_1 + 2x_2 - x_3 + 2x_4 = 0$
$2x_1 + x_2 + x_3 - x_4 = 0$
$3x_1 - x_2 - 2x_3 + 3x_4 = 0$

21. $x_1 + 2x_2 - x_3 = 0$
$x_2 + 2x_3 = 0$
$4x_3 = 0$

22. $x_1 - x_2 = 0$
$3x_1 = 0$
$2x_1 + x_2 = 0$

23. For what value(s) of a does the system have nontrivial solutions?

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -x_1 - x_2 + x_3 &= 0 \\ 3x_1 + 4x_2 + ax_3 &= 0. \end{aligned}$$

24. Consider the system of equations

$$\begin{aligned} x_1 + 3x_2 - x_3 &= b_1 \\ x_1 + 2x_2 \quad\quad &= b_2 \\ 3x_1 + 7x_2 - x_3 &= b_3. \end{aligned}$$

a) Determine conditions on b_1, b_2, and b_3 that are necessary and sufficient for the system to be consistent. [*Hint:* Reduce the augmented matrix for the system.]

b) In each of the following, either use your answer from **a)** to show the system is inconsistent or exhibit a solution.

i) $b_1 = 1, b_2 = 1, b_3 = 3$
ii) $b_1 = 1, b_2 = 0, b_3 = -1$
iii) $b_1 = 0, b_2 = 1, b_3 = 2$

25. Let B be a (4×3) matrix in reduced echelon form.

a) If B has three nonzero rows, then determine the form of B. (Using Fig. 1.5 of Section 1.2 as a guide, mark entries that may or may not be zero by $*$.)

b) Suppose that a system of 4 linear equations in 2 unknowns has augmented matrix A, where A is a (4×3) matrix row equivalent to B. Demonstrate that the system of equations is inconsistent.

In Exercises 26–31, follow the ideas illustrated in Examples 8 and 9 to find the equation of the curve or surface through the given points. For Exercises 28–29, display the graph of the equation as in Fig. 1.7.

26. The line through (3, 1) and (7, 2).

27. The line through (2, 8) and (4, 1).

28. The conic through (−4, 0), (−2, −2), (0, 3), (1, 1), and (4, 0).

29. The conic through (−4, 1), (−1, 2), (3, 2), (5, 1), and (7, −1).

30. The quadric surface through (0, 0, 1), (1, 0, 1), (0, 1, 0), (3, 1, 0), (2, 0, 4), (1, 1, 2), (1, 2, 1), (2, 2, 3), (2, 2, 1).

31. The quadric surface through (1, 2, 3), (2, 1, 0), (6, 0, 6), (3, 1, 3), (4, 0, 2), (5, 5, 1), (1, 1, 2), (3, 1, 4), (0, 0, 2).