Math 544, Final Exam Summer 2007

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are **9** problems **on TWO sides**. The exam is worth a total of 100 points. Problem 1 is worth 20 points. All of the other problems are worth 10 points each. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

You should **KEEP** this copy of your exam.

I will post the solutions on my website later today.

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 2 & 3 & 7 \\ 3 & 4 & 6 & 13 \\ 1 & 2 & 6 & 9 \end{bmatrix}$. Find a basis for the null space of A. Find a basis for the column space of A. Find a basis for the row space of A. Express each column of A as a linear combination of the basis you have chosen for the column space of A. Express each row of A as a linear combination of the basis you have chosen for the row space of A.

2. Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.

3. Let $T\colon V \to W$ be a linear transformation of vector spaces. Let $w_1, \dots, w_b$ be a basis for the image of T; $z_1, \dots, z_a$ be a basis for the null space of T; and $v_1, \dots, v_b$ be vectors in V with $T(v_i) = w_i$ for $1 \le i \le b$. Prove that the vectors $v_1, \dots, v_b, z_1, \dots, z_a$ span V. Recall that the image of T is equal to

$$\{w \in W \mid w = T(v) \text{ for some } v \in V\}.$$

4. Let $U \subseteq V \subseteq W$ be vector spaces. Suppose that v_1, v_2, v_3, v_4 is a basis for W. Suppose further that v_1, v_2, v_3 are in V, but v_4 is not in V. Suppose finally, that v_1 and v_2 are in U, but v_3 and v_4 are not in U. What is the dimension of U? Explain your answer **VERY THOROUGHLY.**

5. Let $V = \{p(x) \in \mathcal{P}_3 \mid p'(1) = 0\}$. Is V a vector space? If yes, then find a basis for V. If no, then show why not? (Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree less than or equal to 3.)

6. Let $V = \{M \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \mid \mathrm{tr}(M) = 0\}$. Is V a vector space? If yes, then find a basis for V. If no, then show why not? (Recall that $\mathrm{Mat}_{3\times 3}(\mathbb{R})$ is the vector space of 3×3 matrices. The trace of the 3×3 matrix

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix}$$

is $\mathrm{tr}(M) = m_{1,1} + m_{2,2} + m_{3,3}$.)

7. Let

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad u_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Express v as a linear combination of u_1, u_2, u_3, u_4. (You are encouraged to notice that u_1, u_2, u_3, u_4 form an orthogonal set of vectors.)

8. Find a matrix B with B^2 equal to $A = \begin{bmatrix} -11 & 15 \\ -20 & 24 \end{bmatrix}$.

9. Find an orthogonal basis for the null space of $A = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}$.