Math 544, Exam 2, Summer 2006 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. **Leave room on the upper left hand corner of each page for the staple.** Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 50 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after class is finished.

1. (10 points) **Let**

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix}.$$

We apply Guassian Elimination to the matrix A. Apply $R_2 \mapsto R_2 - 2R_1$, $R_3 \mapsto R_3 - 2R_1$, and $R_4 \mapsto R_4 - 2R_1$ to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{bmatrix}.$$

Exchange rows 2 and 3 to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{bmatrix}.$$

Apply $R_1 \mapsto R_1 + R_2$ and $R_4 \mapsto R_4 - R_2$ to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

Apply $R_1 \mapsto R_1 + R_3$ and $R_4 \mapsto R_4 - R_3$ to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Multiply rows 2 and 3 by -1 to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the null space of A.

The vectors

$$\boxed{w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

(b) Find a basis for the column space of A.

The vectors

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad A_{*,4} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad A_{*,5} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}}$$

are a basis for the column space of A. Notice that I am writing $A_{*,j}$ for column j of the matrix A.

(c) Find a basis for the row space of A.

The vectors

$$\boxed{\begin{array}{l} z_1 = [1 \quad 2 \quad 3 \quad 0 \quad 0 \quad 1] \\ z_2 = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1] \\ z_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1] \end{array}}$$

are a basis for the row space of A.

(d) Express each column of A in terms of your answer to (b).

We see that

$$\boxed{A_{*,2} = 2A_{*,1}, \quad A_{*,3} = 3A_{*,1}, \quad A_{*,6} = A_{*,1} + A_{*,4} + A_{*,5}.}$$

(e) Express each row of A in terms of your answer to (c).

I write $A_{i,*}$ for row i of A. We see that

$$\boxed{\begin{array}{l} A_{1,*} = z_1 + z_2 + z_3, \\ A_{2,*} = 2z_1 + 2z_2 + z_3, \\ A_{3,*} = 2z_1 + z_2 + 2z_3, \\ A_{4,*} = 2z_1 + z_2 + z_3. \end{array}}$$

2. (8 points) **Find an orthogonal basis for the null space of $A = [1 \quad 1 \quad 1 \quad 2]$.**

We start with the basis

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for the null space of A. Let $u_1 = v_1$. Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = 2u_2' = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}.$$

We check that $Au_2 = 0$ and $u_1^{\mathrm{T}} u_2 = 0$. Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-2}{3} \\ \frac{-2}{3} \\ \frac{-2}{3} \\ 1 \end{bmatrix}.$$

Let

$$u_3 = 3u_3' = \begin{bmatrix} -2 \\ -2 \\ -2 \\ 3 \end{bmatrix}.$$

We check that $Au_3 = 0$, $u_1^{\mathrm{T}} u_3 = 0$, and $u_2^{\mathrm{T}} u_3 = 0$. We conclude that

$$\boxed{u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -2 \\ -2 \\ -2 \\ 3 \end{bmatrix}}$$

is an orthogonal basis for the null space of A.

3. (8 points) **Let A and B be $n \times n$ matrices with A non-singular. For each question: If the answer is yes, then PROVE the assertion. If the answer is no, then give a COUNTER EXAMPLE.**

   (a) **Does the null space of BA have to equal the null space of B ?**

NO. Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We see that $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The matrices B and AB have different null spaces. For example $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the null space of B, but not the null space of BA. On the other hand, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the null space of BA but not the null space of B.

   (b) **Does the column space of BA have to equal the column space of B ?**

YES. The column space of BA is contained in the column space of B because every vector in the column space of BA has the form BAv for some vector v. It is clear that $BAv = B(Av)$ which is B times a vector and therefore in the column space of B.

The fact that the column space of B is contained in the column space of BA uses the hypothesis that A is non-singular. A typical element of the column space of B has the form Bu for some vector u. We see that $Bu = BA(A^{-1}u)$ and this vector has the form BA times some vector; so this vector is also in the column space of BA.

   (d) **Does the the dimension of column space of BA have to equal the the dimension of column space of B ?**

YES. The matrices B and BA have the SAME column space. This one column space has a dimension.

   (c) **Does the dimension of the null space of BA have to equal the dimension of the null space of B ?**

YES. The nullity of B is equal to the number of columns of B minus the rank of B. The nullity of BA is equal to the number of columns of BA minus the rank of BA. The matrices B and BA have the same rank by part (d). The matrix A

is square; so, B and BA have the same number of columns. We conclude that B and BA have the same nullity.

4. (3 points) **Define "null space". Use complete sentences. Include everything that is necessary, but nothing more.**

The *null space* of the matrix A is the set of all vectors v such that $Av = 0$.

5. (3 points) **Define "column space". Use complete sentences. Include everything that is necessary, but nothing more.**

The *column space* of the matrix A is the set of all vectors Av for some vector v.

6. (3 points) **Define "dimension". Use complete sentences. Include everything that is necessary, but nothing more.**

The *dimension* of the vector space V is the number of vectors in a basis for V.

7. (5 points) **Suppose $U \subseteq V$ are vector spaces with the same finite dimension. Does U have to equal V? If the answer is yes, then PROVE the assertion. If the answer is no, then give a COUNTER EXAMPLE.**

YES. Let $n = \dim U = \dim V$. Let $u_1, \dots, u_n$ be a basis for U. The vectors $u_1, \dots, u_n$ are linearly independent vectors in V. One of the dimension theorems tells us that every linearly independent set of vectors in the vector space V is contained in a basis for V. The first Theorem about vector space dimension tells us that every basis for V has n vectors. It is not possible to adjoin extra vectors to the set $u_1, \dots, u_n$ in order to produce a basis for V. The only remaining option is that $u_1, \dots u_n$ is already a basis for V. In other words, $u_1, \dots u_n$ already span all of V and every vector in V is already in U.

8. (5 points) **Let v_1, v_2, v_3 be linearly independent vectors in $\mathbb{R}^3$. Can every vector in $\mathbb{R}^3$ be written in terms of v_1, v_2, v_3 in a unique way? If the answer is yes, then PROVE the assertion. If the answer is no, then give a COUNTER EXAMPLE.**

YES. The three linearly independent vectors v_1, v_2, v_3 in the 3-dimensional vector space $\mathbb{R}^3$ are automatically a basis for $\mathbb{R}^3$. It is always true that if $v_1, \dots, v_n$ is a basis for a vector space V, then every vector in V may be written in terms of the basis $v_1, \dots, v_n$ in a unique manner.

9. (5 points) **Let** v_1, v_2, v_3, v_4 **be vectors in** $\mathbb{R}^4$**. Suppose that** v_1, v_2, v_3 **are linearly independent;** v_1, v_2, v_4 **are linearly independent;** v_1, v_3, v_4 **are linearly independent; and** v_2, v_3, v_4 **are linearly independent. Do** v_1, v_2, v_3, v_4 **have to be linearly independent? If the answer is yes, then PROVE the assertion. If the answer is no, then give a COUNTER EXAMPLE.**

NO. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

It is clear that v_1, v_2, v_3 are linearly independent; v_1, v_2, v_4 are linearly independent; v_1, v_3, v_4 are linearly independent; v_2, v_3, v_4 are linearly independent; but v_1, v_2, v_3, v_4 are linearly dependent.