Math 544, Final Exam , Summer 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 11 problems. Problem 1 is worth 20 points. Each of the other problems is worth 8 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

I will e-mail your grade to you.

I will post the solutions on my website shortly after the class is finished.

1. Let
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$
   (a) Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.
   (b) Find the general solution of $Ax = c$. List three specific solutions, if possible. Check your solutions.
   (c) Find a basis for the null space of A.
   (d) Find a basis for the column space of A.
   (e) Find a basis for the row space of A.
   (f) Express each column of A in terms of your answer to (d).
   (g) Express each row of A in terms of your answer to (e).

2. Let $U \subseteq V$ be vector spaces. Is it always true that $\dim U \le \dim V$? If yes, prove your answer. If no, give an example.

3. Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation. Suppose v_1, v_2, and v_3 are linearly independent in V. Do $T(v_1)$, $T(v_2)$, and $T(v_3)$ have to be linearly independent in W? If yes, prove your answer. If no, give an example.

4. Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation. Suppose v_1, v_2, and v_3 are vectors in V and $T(v_1)$, $T(v_2)$, and $T(v_3)$ are linearly independent in W. Do v_1, v_2, and v_3 have to be linearly independent in V? If yes, prove your answer. If no, give an example.

5. Let A be an $n \times n$ matrix. Let v_1 and v_2 be non-zero vectors in $\mathbb{R}^n$ with $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, where λ_1 and λ_2 are distinct real numbers. Prove that v_1 and v_2 are linearly independent.

6. Let $A = \begin{bmatrix} 1 & -1 & -1 & -2 \\ 1 & 1 & -1 & -2 \\ 1 & 0 & 2 & -2 \\ 2 & 0 & 0 & 3 \end{bmatrix}$. Find the inverse of A. You may do the problem any way you like; however, you might want to notice that the columns of A form an orthogonal set.

7. Let $A = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ -\frac{3}{2} & \frac{7}{4} \end{bmatrix}$. Find $\lim\limits_{n\to\infty} A^n$.

8. Find a basis for the vector space spanned by

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}.$$

9. The *trace* of the square matrix M is the sum of the elements on the main diagonal of M. Let V be the vector space of all 3×3 matrices M with the trace of M equal to zero. Find a basis for V.

10. Recall that P_4 is the vector space of all polynomials of degree less than or equal to four. Let W be the subspace of all polynomials in P_4 which satisfy $p(1) + p(-1) = 0$ and $p(2) + p(-2) = 0$. What is the dimension of W?

11. Let $u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Find vectors u_2, u_3, and u_4 in $\mathbb{R}^4$ so that u_1, u_2, u_3, u_4 is an orthogonal set.