Math 544, Exam 1, Summer 2005 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 6 problems. Problem 1 is worth 10 points. Each of the other problems is worth 8 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 5 \\ 1 & 4 & 5 & 2 & 1 & 8 \\ 1 & 4 & 5 & 2 & 2 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

We apply row operations to

$$\left[\begin{array}{cccccc|c} 1 & 4 & 5 & 1 & 1 & 5 & 4 \\ 1 & 4 & 5 & 2 & 1 & 8 & 3 \\ 1 & 4 & 5 & 2 & 2 & 8 & 5 \end{array}\right].$$

Apply $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - R_1$ to get:

$$\left[\begin{array}{cccccc|c} 1 & 4 & 5 & 1 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 - R_2$ and $R_3 \mapsto R_3 - R_2$ to get

$$\left[\begin{array}{cccccc|c} 1 & 4 & 5 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{array}\right].$$

Apply $R_1 \mapsto R_1 - R_3$ to get

$$\left[\begin{array}{cccccc|c} 1 & 4 & 5 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{array}\right].$$

The general solution of the system of equations is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \\ \text{where } x_2,\ x_3, \text{ and } x_6 \text{ may be any real number.}}$$

- When $x_2 = x_3 = x_6 = 0$, the specific solution

$$v_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

is attained. We check that

$$Av_1 = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 5 \\ 1 & 4 & 5 & 2 & 1 & 8 \\ 1 & 4 & 5 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-1+2 \\ 3-2+2 \\ 3-2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = b. \checkmark$$

- When $x_2 = 1$, $x_3 = x_6 = 0$, the specific solution

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

is attained. We check that

$$Av_2 = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 5 \\ 1 & 4 & 5 & 2 & 1 & 8 \\ 1 & 4 & 5 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+4-1+2 \\ -1+4-2+2 \\ -1+4-2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = b. \checkmark$$

- When $x_3 = 1$, $x_2 = x_6 = 0$, the specific solution

$$v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

is attained. We check that

$$Av_3 = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 5 \\ 1 & 4 & 5 & 2 & 1 & 8 \\ 1 & 4 & 5 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+5-1+2 \\ -2+5-2+2 \\ -2+5-2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = b. \checkmark$$

- When $x_6 = 1$, $x_2 = x_3 = 0$, the specific solution

$$v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -4 \\ 2 \\ 1 \end{bmatrix}$$

is attained. We check that

$$Av_4 = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 5 \\ 1 & 4 & 5 & 2 & 1 & 8 \\ 1 & 4 & 5 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-4+2+5 \\ 1-8+2+8 \\ 1-8+4+8 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = b. \checkmark$$

2. **Consider the system of linear equations.**

$$\begin{aligned} x_1 + 4ax_2 &= 4 \\ ax_1 + x_2 &= 2. \end{aligned}$$

(a) **Which values for a cause the system to have no solution?**
(b) **Which values for a cause the system to have exactly one solution?**
(c) **Which values for a cause the system to have an infinite number of solutions?**

Explain thoroughly.

Apply row opperations to

$$\left[\begin{array}{cc|c} 1 & 4a & 4 \\ a & 1 & 2 \end{array}\right]$$

Apply $R_2 \mapsto R_2 - aR_1$ to get

$$\left[\begin{array}{cc|c} 1 & 4a & 4 \\ 0 & 1-4a^2 & 2-4a \end{array}\right]$$

If $1-4a^2 \neq 0$, then the system of equations has a unique solution.
If $1-4a^2 = 0$ and $2-4a = 0$, then the system of equations has infinitely many solutions.
If $1-4a^2 = 0$ and $2-4a \neq 0$, then the system of equations has no solution.

We notice that $1-4a^2=0$ when $(1-2a)(1+2a)=0$. Thus, $a=\frac{1}{2}$, or $a=-\frac{1}{2}$.
We notice that when $a=\frac{1}{2}$, then $2-4a=0$
We notice that when $a=-\frac{1}{2}$, then $2-4a\neq 0$.
We conclude

If a is different than $\frac{1}{2}$ and $-\frac{1}{2}$, then the system has a unique solution. If $a=\frac{1}{2}$, then the system has infinitely many solutions. If $a=-\frac{1}{2}$, then the system has no solution.

3. **Are the vectors**

$$v_1=\begin{bmatrix}1\\2\\3\end{bmatrix},\quad v_2=\begin{bmatrix}4\\12\\18\end{bmatrix},\quad v_3=\begin{bmatrix}1\\4\\6\end{bmatrix}$$

linearly independent? Explain thoroughly.

These vectors are $\boxed{\text{linearly dependent}}$, because $v_2=2v_1+2v_3$. One might solve $c_1v_1+c_2+v_2+c_3v_3=0$. Apply row operations to

$$\begin{bmatrix}1&4&1\\2&12&4\\3&18&6\end{bmatrix}.$$

Apply $R_2\mapsto R_2-2R_1$ and $R_3\mapsto R_3-3R_1$ to get:

$$\begin{bmatrix}1&4&1\\0&4&2\\0&6&3\end{bmatrix}.$$

Apply $R_2\mapsto \frac{1}{4}R_2$ to get

$$\begin{bmatrix}1&4&1\\0&1&\frac{1}{2}\\0&6&3\end{bmatrix}.$$

Apply $R_3\mapsto R_3-6R_2$ and $R_1\mapsto R_1-4R_2$ to get

$$\begin{bmatrix}1&0&-1\\0&1&\frac{1}{2}\\0&0&0\end{bmatrix}.$$

So c_3 is a free variable, $c_1=c_3$, and $c_2=-(\frac{1}{2})c_3$. In particular, when $c_3=2$, then we have $c_1=2$ and $c_2=-1$. In other words, $2v_1-v_2+2v_3=0$.

4. **(True or False. If true, PROVE the result. If false, give a counter EXAMPLE.) If A is a 2×2 matrix with $A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}$, then $A=\begin{bmatrix}0&0\\0&0\end{bmatrix}$.**

$\boxed{\text{False.}}$ If $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, then $A\neq 0$, but

$$A^2=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

5. **Let A and B be symmetric $n \times n$ matrices. Suppose that AB is also a symmetric matric. Prove that $AB = BA$.**

When all of the listed hypotheses hold, then we have

$$AB = (AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}} = BA.$$

The first equality holds because AB is a symmetric matrix. The second equality holds for all matrices – we proved this result in class. The last equality holds because B and A both are symmetric matrices.

6. **Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^5$. Suppose that v_1, v_2, v_3 are linearly dependent. Do the vectors v_1, v_2, v_3, v_4 HAVE to be linearly dependent? If yes, PROVE the result. If no, show an EXAMPLE.**

$\boxed{\text{Yes.}}$ We are told that there are numbers c_1, c_2, c_3, not all zero, with $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. Take the old numbers c_1, c_2, c_3 together with $c_4 = 0$. We now have $c_1 v_1 + c_2 v_2 + c_3 v_3 + 0 v_4 = 0$, and at least one of the coefficients is non-zero. The vectors v_1, v_2, v_3, v_4 are linearly dependent.