Math 544, Final Exam, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 14 problems. Problem 1 is worth 22 points. Each of the rest of the problems is worth 6 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your course grade from VIP.

I will post the solutions on my website shortly after the exam is finished.

1. **Let**

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \textbf{and} \quad c = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

(a) **Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.**
(b) **Find the general solution of $Ax = c$. List three specific solutions, if possible. Check your solutions.**
(c) **Find a basis for the null space of A.**
(d) **Find a basis for the column space of A.**
(e) **Find a basis for the row space of A.**
(f) **Express each column of A in terms of your answer to (d).**
(g) **Express each row of A in terms of your answer to (e).**

We study the augmented matrix

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 2 & 4 & 6 & 2 & 1 & 5 & 2 & 2 \\ 2 & 4 & 6 & 1 & 2 & 5 & 2 & 2 \\ 2 & 4 & 6 & 1 & 1 & 4 & 2 & 3 \end{array}\right].$$

Apply $R_2 \mapsto R_2 - 2R_1$, $R_3 \mapsto R_3 - 2R_1$, and $R_4 \mapsto R_4 - 2R_1$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right].$$

Exchange rows 2 and 3 to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 + R_2$ and $R_4 \mapsto R_4 - R_2$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 + R_3$ and $R_4 \mapsto R_4 - R_3$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Multiply rows 2 and 3 by -1 to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

The general solution to $Ax = b$ is

$$\textbf{(a)} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{for any } x_2,\, x_3,\, x_6 \text{ in } \mathbb{R}.$$

Four specific solutions are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

(I obtained v_1 by setting $x_2 = x_3 = x_6 = 0$; v_2 by setting $x_2 = 1, x_3 = x_6 = 0$; v_3 by setting $x_3 = 1, x_2 = x_6 = 0$; and v_4 by setting $x_6 = 1, x_2 = x_3 = 0$.) I check that

$$Av_1 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_4 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b. \checkmark$$

(b) The equations $Ax = c$ have $\boxed{\text{NO solution.}}$

(c) The vectors

$$\boxed{w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

(d) The vectors

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad A_{*,4} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad A_{*,5} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}}$$

are a basis for the column space of A.

(e) The vectors

$$\boxed{\begin{aligned} z_1 &= \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \\ z_2 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \\ z_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}}$$

are a basis for the row space of A.

(f)

$$\boxed{A_{*,2} = 2A_{*,1}, \quad A_{*,3} = 3A_{*,1}, \quad A_{*,6} = A_{*,1} + A_{*,4} + A_{*,5}.}$$

(g)

$$\boxed{\begin{aligned} A_{1,*} &= z_1 + z_2 + z_3, \\ A_{2,*} &= 2z_1 + 2z_2 + z_3, \\ A_{3,*} &= 2z_1 + z_2 + 2z_3, \\ A_{4,*} &= 2z_1 + z_2 + z_3. \end{aligned}}$$

2. **Find an orthogonal basis for the null space of the matrix A from problem 1.**

The vectors

$$w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

are a basis for the null space of A. Let

$$u_1 = w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$u_2' = w_2 - \frac{u_1^{\mathrm{T}} w_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = 5u_2' = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$u_3' = w_3 - \frac{u_1^{\mathrm{T}} w_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} w_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{70} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{70}\left(70\begin{bmatrix}-1\\0\\0\\-1\\-1\\1\end{bmatrix} - 28\begin{bmatrix}-2\\1\\0\\0\\0\\0\end{bmatrix} - 3\begin{bmatrix}-3\\-6\\5\\0\\0\\0\end{bmatrix}\right) = \frac{1}{70}\begin{bmatrix}-70+56+9\\-28+18\\-15\\-70\\-70\\70\end{bmatrix} = \frac{1}{70}\begin{bmatrix}-5\\-10\\-15\\-70\\-70\\70\end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix}-1\\-2\\-3\\-14\\-14\\14\end{bmatrix}.$$

We let $u_3 = 14u_3'$. Our answer is

$$\boxed{u_1 = \begin{bmatrix}-2\\1\\0\\0\\0\\0\end{bmatrix}, \quad u_2 = \begin{bmatrix}-3\\-6\\5\\0\\0\\0\end{bmatrix}, \quad u_3 = \begin{bmatrix}-1\\-2\\-3\\-14\\-14\\14\end{bmatrix}.}$$

We check that $u_1^{\mathrm{T}}u_2 = u_1^{\mathrm{T}}u_3 = u_2^{\mathrm{T}}u_3 = 0$, and $Au_i = 0$ for each i.

3. **Find a matrix B with $B^2 = A$, where $A = \begin{bmatrix}-5 & -9\\6 & 10\end{bmatrix}$.**

We find the eigenvalues of A:

$$0 = \det(A-\lambda I) = \det\begin{bmatrix}-5-\lambda & -9\\6 & 10-\lambda\end{bmatrix} = (-5-\lambda)(10-\lambda)+54$$

$$= 4 - 5\lambda + \lambda^2 = (\lambda-1)(\lambda-4).$$

The eigenvalues of A are 1 and 4.

We find the eigenvectors of A that belong to $\lambda = 1$: We solve $Av = v$ or $(A-I)v = 0$. So we apply Guassian elimination to

$$A - I = \begin{bmatrix}-6 & -9\\6 & 9\end{bmatrix}.$$

Apply $R_2 \mapsto R_2 + R_1$ to obtain

$$\begin{bmatrix}-6 & -9\\0 & 0\end{bmatrix}.$$

Divide row 1 by -6:

$$\begin{bmatrix}1 & 3/2\\0 & 0\end{bmatrix}.$$

One basis for the eigenspace which belongs to $\lambda = 1$ is $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$. Another basis for the eigenspace belonging to $\lambda = 1$ is $v_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. (Check that $Av_1 = v_1$.)

We find the eigenvectors of A that belong to $\lambda = 4$: We solve $Av = 4v$ or $(A - 4I)v = 0$. So we apply Guassian elimination to

$$A - 4I = \begin{bmatrix} -9 & -9 \\ 6 & 6 \end{bmatrix}.$$

Divide row 1 by -9, to obtain

$$\begin{bmatrix} 1 & 1 \\ 6 & 6 \end{bmatrix}.$$

Apply $R_2 \mapsto R_2 - 6R_1$ to obtain

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The vector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ belongs to $\lambda = 4$.

We see that:

$$A \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let $S = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. We see that S is non-singular. We have $A = SDS^{-1}$. We see that $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. If we let

$$B = S \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} S^{-1},$$

then B will square to become A. So

$$B = S \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} S^{-1} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}.}$$

Do check that $B^2 = A$. (There are other correct final answers.)

4. **Define "linearly dependent". Use complete sentences.**

The vectors $v_1, \dots, v_n$ of the vector space V are linearly dependent if there exist real numbers $c_1, \dots, c_n$, not all of which are zero, with $\sum_{i=1}^{n} c_i v_i = 0$.

5. **Define "linear transformation". Use complete sentences.**

The function T from the vector space V to the vector space W is a linear transformation if $T(v_1+v_2)=T(v_1)+T(v_2)$ and $T(cv_1)=cT(v_1)$ for all v_1 and v_2 in V and all $c\in\mathbb{R}$.

6. **Define "null space". Use complete sentences.**

The null space of the matrix A is the set of all column vectors x with $Ax=0$.

7. **Let $T\colon \mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation which fixes the origin and rotates the xy-plane counter-clockwise by 45 degrees. Find a matrix M with $T(v)=Mv$ for all vectors v in $\mathbb{R}^2$.**

$$\boxed{M=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}.}$$

8. **Let A be a matrix, v_1 and v_2 be non-zero vectors, and λ_1 and λ_2 be real numbers. Suppose that $Av_1=\lambda_1 v_1$, $Av_2=\lambda_2 v_2$, and $\lambda_1\neq\lambda_2$. PROVE that v_1 and v_2 are linearly independent.**

Suppose

(*) $$c_1v_1+c_2v_2=0.$$

Multiply (*) by A to see that

(**) $$c_1\lambda_1v_1+c_2\lambda_2v_2=0.$$

Multiply (*) by λ_1 to see that

(***) $$c_1\lambda_1v_1+c_2\lambda_1v_2=0.$$

Subtract (**) minus (***) to see that every entry of the column vector

$$c_2(\lambda_2-\lambda_1)v_2$$

is zero. The hypothesis tells us that $\lambda_2-\lambda_1$ is not zero. The hypothesis also tells us that some entry of v_2 is not zero. We conclude that $c_2=0$. Now $c_1v_1=0$, with $v_1\neq 0$. Again it follows that c_1 must also be zero. The only numbers c_1 and c_2 with (*) are $c_1=c_2=0$. The vectors v_1 and v_2 are linearly independent.

9. **Let $T\colon V\to W$ be a linear transformation, and let v_1, v_2, and v_3 be vectors in V, with $T(v_1)$, $T(v_2)$, and $T(v_3)$ linearly independent. Do the vectors v_1, v_2, and v_3 have to be linearly independent? If yes, PROVE the result. If no, give a counter EXAMPLE.**

$\boxed{\text{YES}}$ If

(★) $$c_1v_1+c_2v_2+c_3v_3=0,$$

then apply the linear transformation T to both sides of (★) to see that

$$T(c_1v_1 + c_2v_2 + c_3v_3) = 0.$$

Use the defining properties of linear transformation to see that

$$(\bigstar\bigstar) \qquad c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = 0.$$

The hypothesis tells us that the vectors $T(v_1)$, $T(v_2)$, and $T(v_3)$ linearly independent. The only constants c_1, c_2, and c_3 for which (★★) holds are $c_1 = c_2 = c_3 = 0$. We now know that the only constants for which (★) are $c_1 = c_2 = c_3 = 0$. We conclude that v_1, v_2, and v_3 are linearly independent.

10. **Let V be the vector space of all polynomials $p(x)$ of degree three or less which have the property that $p(2) = 0$ and $p'(2) = 0$. Find a basis for V. Explain.**

One basis is $\boxed{p_1 = (x-2)^2, p_2 = (x-2)^3}$. It is easy to p_1 and p_2 are linearly independent elements of V. I finish my argument by showing that $\dim V \le 2$. Well, $V \subsetneq W \subsetneq \mathcal{P}_3$, where W is the subspace of all polynomials $p(x)$ of degree three or less which have the property that $p(2) = 0$ and $\mathcal{P}_3$ is the vector space of all polynomials $p(x)$ of degree three or less. I know that $W \ne \mathcal{P}_3$ because $x \in \mathcal{P}_3$, but $x \notin W$. I know that $V \ne W$ because $x - 2 \in W$ but $x - 2 \notin V$. Thus, $\dim V < \dim W < \dim \mathcal{P}_3 = 4$, and $\dim V \le 2$, as desired.

11. **Let V be the vector space of all differentiable real-valued functions which are defined on all of $\mathbb{R}$. Let W be the vector space of all real-valued functions which are defined on all of $\mathbb{R}$. Let T from V to W be the function which is given by $T(f(x)) = f'(x)$. Is T a linear transformation? Explain very thoroughly.**

$\boxed{\text{YES}}$ We learned in calculus that $(f+g)' = f' + g'$. We also learned in calculus that $(rf)' = rf'$.

12. **Let A and B be $n \times n$ matrices. Is the null space of B contained in the null space of AB? If yes, PROVE the result. If no, give a counter EXAMPLE.**

$\boxed{\text{YES}}$ Let v be a vector in the null space of B. So $Bv = 0$. So $ABv = 0$. So v is in the null space of AB.

13. **Let A and B be $n \times n$ matrices. Is the column space of B contained in the column space of AB? If yes, PROVE the result. If no, give a counter EXAMPLE.**

$\boxed{\text{NO}}$. Consider $B = I$ and $A = 0$. The column space of B is all of $\mathbb{R}^n$. The column space of $AB = 0$ is the zero vector. It is not true that all of $\mathbb{R}^n$ is contained in the zero vector.

14. **Let**

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let V be a subspace of $\mathbb{R}^4$. Suppose that $v_1 \in V$, $v_2 \in V$, $v_3 \notin V$, and $v_4 \notin V$. Do you have enough information to determine the dimension of V? Explain very thoroughly.

$\boxed{\text{NO}}$. The vector space V could have dimension 2. (In this case v_1 and v_2 are a basis for V.) On the other hand, the vector space V could have dimension 3. For example, the vector space V spanned by v_1, v_2, and

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

has dimension 3 and does not contain v_3 or v_4.