Math 544, Exam 2 Solutions, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet; start each computational problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

There are 10 problems. Each problem is worth 5 points. The exam is worth a total of 50 points. SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions. CHECK that the specific solutions satisfy the equations.**

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}.$$

Apply the row operations $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - 2R_1$ to the matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 10 & 5 \\ 1 & 2 & -1 & 2 & 16 & 5 \\ 2 & 4 & -2 & 3 & 26 & 10 \end{array}\right]$$

to obtain

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 10 & 5 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{array}\right].$$

Apply $R_3 \mapsto R_3 - R_2$ and $R_1 \mapsto R_1 - R_2$ to get

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 4 & 5 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution of the system of equations is

$$\boxed{\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ 0 \\ -6 \\ 1 \end{bmatrix} \;\middle|\; x_2, x_3, x_5 \in \mathbb{R} \right\}}.$$

Some specific solutions of this system of equations are

$$v_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -6 \\ 1 \end{bmatrix}.$$

(In v_1, I took $x_2 = x_3 = x_5 = 0$. In v_2, I took $x_2 = 1$, $x_3 = 0$, and $x_5 = 0$. In v_3, I took $x_2 = 0$, $x_3 = 1$, and $x_5 = 0$. In v_4, I took $x_2 = 0$, $x_3 = 0$, and $x_5 = 1$.) We check

$$Av_1 = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 \\ 5 \cdot 1 \\ 5 \cdot 2 \end{bmatrix} = b. \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \cdot 1 + 2 \cdot 1 \\ 3 \cdot 1 + 2 \cdot 1 \\ 3 \cdot 2 + 1 \cdot 4 \end{bmatrix} = b. \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 1 \cdot (-1) \\ 6 \cdot 1 + 1 \cdot (-1) \\ 6 \cdot 2 + 1 \cdot (-2) \end{bmatrix} = b. \checkmark$$

$$Av_4 = \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 - 6 \cdot 1 + 1 \cdot 10 \\ 1 \cdot 1 - 6 \cdot 2 + 1 \cdot 16 \\ 1 \cdot 2 - 6 \cdot 3 + 1 \cdot 26 \end{bmatrix} = b. \checkmark$$

2. **Express** $a = \begin{bmatrix} 11 \\ 16 \end{bmatrix}$ **as a linear combination of** $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **and** $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$**.**

We want to find c_1 and c_2 with

$$c_1 v_1 + c_2 v_2 = a. \tag{1}$$

This amounts to solving a system of equations. Apply $R_2 \mapsto R_2 - 2R_1$ to the matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 11 \\ 2 & 4 & 16 \end{array}\right]$$

to obtain

$$\left[\begin{array}{cc|c} 1 & 3 & 11 \\ 0 & -2 & -6 \end{array}\right].$$

Divide row 2 by -2 to obtain

$$\left[\begin{array}{cc|c} 1 & 3 & 11 \\ 0 & 1 & 3 \end{array}\right]$$

Apply $R_1 \mapsto R_1 - 3R_2$ to obtain

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Thus, the solution of (1) is $c_1 = 2$ and $c_2 = 3$. Observe that

$$2v_1 + 3v_2 = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+9 \\ 4+12 \end{bmatrix} = a,$$

as expected; so the answer is $\boxed{a = 2v_1 + 3v_2}$.

3. **Are the vectors** $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$**,** $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$**, and** $v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ **linearly independent or linearly dependent? Explain.**

We solve the system of equations

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0. \tag{2}$$

Apply $R_2 \mapsto R_2 - 2R_1$ and $R_3 \mapsto R_3 - 3R_1$ to

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

to obtain

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

Divide row 2 by -3 to see that

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix}.$$

Apply $R_3 \mapsto R_3 + 6R2$ and $R_1 \mapsto R_1 - 4R2$ to obtain

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The general solution of (2) is $c_1 = c_3$, $c_2 = -2c_3$ and c_3 is a free variable. In particular, if we take $c_3 = 1$, then we have $c_1 = 1$, $c_2 = -2$, and $c_3 = 1$. It is indeed true that $1v_1 - 2v_2 + 1v_3 = 0$ because $1 - 8 + 7 = 0$, $2 - 10 + 8 = 0$ and $3 - 12 + 9 = 0$.

$$\boxed{\text{The vectors } v_1,\, v_2,\, v_3 \text{ are linearly DEPENDENT.}}$$

4. **The vectors v_1 , v_2 , and v_3 are linearly independent. Do the vectors $v_1 - v_2$, $v_2 - v_3$, and $v_3 - v_1$ have to be linearly independent? If yes, then prove the result. If no, then give an example.**

$\boxed{\text{NO.}}$ In fact, the vectors $v_1 - v_2$, $v_2 - v_3$, and $v_3 - v_1$ are ALWAYS linearly DEPENDENT because

$$1(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_1) = 0$$

for EVERY choice of v_1 , v_2 , and v_3 . If you want a concrete example, the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent; but the vectors

$$w_1 = v_1 - v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad w_2 = v_2 - v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad w_3 = v_3 - v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are linearly dependent since

$$1w_1 + 1w_2 + 1w_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is most likely that the answer to this problem did not just jump into your head. You probably set out to decide if $v_1 - v_2$, $v_2 - v_3$, and $v_3 - v_1$ are linearly independent. So you set out to solve

$$c_1(v_1 - v_2) + c_2(v_2 - v_3) + c_3(v_3 - v_1) = 0. \tag{3}$$

Equation (3) is equivalent to

$$(c_1 - c_3)v_1 + (c_2 - c_1)v_2 + (c_3 - c_2)v_3 = 0. \tag{4}$$

The vectors v_1 , v_2 , and v_3 are linearly independent; consequently the coefficients which appear in (4) MUST be zero; that is,

$$\begin{cases} c_1 - c_3 = 0 \\ c_2 - c_1 = 0 \\ c_3 - c_2 = 0 \end{cases} \tag{5}$$

We know how to solve (5). Apply $R_2 \mapsto R_2 + R_1$ to

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

to obtain

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Apply $R_3 \mapsto R_3 + R_2$ to obtain

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution to (5) is $c_1 = c_3$, $c_2 = c_3$ and c_3 is a free variable. If we take $c_3 = 1$, then $c_1 = c_2 = c_3 = 1$ is a solution of (5). Hence $c_1 = c_2 = c_3 = 1$ is a solution of (3), and that is where my answer began.

5. **Let A be an $n \times n$ matrix with the property that $Ax = b$ has a unique solution for every vector b in $\mathbb{R}^n$. Does $A^{\mathrm{T}}x = b$ have to have a unique solution for every vector b in $\mathbb{R}^n$? If yes, then prove the result. If no, then give an example.**

YES The hypothesis tells us that every condition of the non-singular matrix theorem holds for the matrix A. In particular, the matrix A has an inverse. It follows that the matrix A^{T} has an inverse. (Indeed, the inverse of A^{T} is merely the transpose of A^{-1}, as we saw in class.) Every condition in the non-singular matrix theorem holds for the matrix A^{T}. In particular, the system of equations $A^{\mathrm{T}}x = b$ has a unique solution for every vector b in $\mathbb{R}^n$.

6. **Define "linearly independent". Use complete sentences.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^m$ are *linearly independent* if the ONLY numbers $c_1, \dots, c_p$, with $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

7. **Define "non-singular". Use complete sentences.**

The square matrix A is *non-singular* if the only column vector x with $Ax = 0$ is $x = 0$.

8. **State the result about the linear dependence of p vectors in m-space. (I call this the Short Fat Theorem).**

If $p > m$, then any list of p vectors from $\mathbb{R}^m$ is linearly DEPENDENT.

9. **Let** $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_1+2x_2+3x_3=0 \\ 2x_1+4x_2+6x_3=0 \\ x_1-7x_2+9x_3=0 \end{matrix} \right\}$. **Is W a vector space? Explain.**

YES. The set W is the null space of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & -7 & 9 \end{bmatrix}$. We proved in class that the null space of any matrix is a vector space.

10. **Let** $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, 0 \leq x_1 x_2 \right\}$**. Is** W **a vector space? Explain.**

$\boxed{\text{NO.}}$ The set W is not closed under addition because $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ are in W, but the sum $v_1 + v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is not in W.