Math 544 Final Exam Summer 2003 Solutions

PRINT Your Name:________________________________

There are 17 problems on 10 pages. Problem 2 is worth 10 points. Problem 17 is worth 15 points. Each of the other problems is worth 5 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your course grade to you. If I don't already know your e-mail address and you want me to know it, send me an e-mail. Otherwise, get your course grade from VIP.

The solutions will be posted at my website shortly after the exam is finished.

1. **Define "basis". Use complete sentences.**

The vectors $v_1, \dots, v_n$ are a basis for the vector space V if $v_1, \dots, v_n$ are linearly independent and span V.

2. **State any TWO of the four theorems about dimension. Use complete sentences.**

Theorem 1. If V is a subsapce of $\mathbb{R}^n$, then every basis for V has the same number of vectors.

Theorem 2. If V is a subsapce of $\mathbb{R}^n$, then every linearly independent subset in V is part of a basis for V.

Theorem 3. If V is a subsapce of $\mathbb{R}^n$, then every finite spanning set for V contains a basis for V.

Theorem 4. If A is a matrix, then the dimension of the column space of A plus the dimension of the null space of A is equal to the number of columns of A.

3. **Give an example of a 4×3 matrix A of rank 2. (Recall that the rank of a matrix is the dimension of its column space.)**

(a) For your matrix A, which vectors b have the property that $Ax = b$ has a solution.
(b) For your matrix A, give an example of a non-zero vector b for which $Ax = b$ DOES have a solution.
(c) For your matrix A, give an example of a non-zero vector b for which $Ax = b$ does NOT have a solution.

My matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) The equation $Ax = b$ has a solution if and only if the vector b has the form $b = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{bmatrix}$ for some numbers b_1 and b_2.

(b) If $b = \begin{bmatrix} 1 \\ 12 \\ 0 \\ 0 \end{bmatrix}$, then $Ax = b$ DOES have a solution.

(c) If $b = \begin{bmatrix} 0 \\ 1 \\ 12 \\ 0 \end{bmatrix}$, then $Ax = b$ does NOT have a solution.

4. **Let W be the set of all polynomials $f(x)$ of degree less than or equal to 3 with $f(3) = 0$. Is W a vector space? If YES, then give a BASIS for W, no proof is needed. If NO, give an EXAMPLE which shows that W is not closed under addition or scalar multiplication.**

$\boxed{\text{YES}}$, W is a vector space with basis

$$\boxed{x - 3, \quad x^2 - 9, \quad x^3 - 27}.$$

5. **Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation**

with

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\\3\end{bmatrix}\quad \textbf{and}\quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\3\\1\end{bmatrix}.$$

Find $T\left(\begin{bmatrix}5\\3\end{bmatrix}\right)$**.**

I see that $\begin{bmatrix}5\\3\end{bmatrix}=3\begin{bmatrix}1\\1\end{bmatrix}+2\begin{bmatrix}1\\0\end{bmatrix}$. It follows that

$$T\left(\begin{bmatrix}5\\3\end{bmatrix}\right)=T\left(3\begin{bmatrix}1\\1\end{bmatrix}+2\begin{bmatrix}1\\0\end{bmatrix}\right)=3T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)+2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$=3\begin{bmatrix}2\\3\\1\end{bmatrix}+2\begin{bmatrix}1\\2\\3\end{bmatrix}=\boxed{\begin{bmatrix}8\\13\\9\end{bmatrix}}.$$

6. **Let W be the set of 2×2 singular matrices. Is W a vector space? If YES, then give a BASIS for W, no proof is needed. If NO, give an EXAMPLE which shows that W is not closed under addition or scalar multiplication.**

$\boxed{\text{NO}}$, W is a not vector space. The matrices

$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix}\qquad B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

are both in W, but the sum $A+B$, which is the identity matrix, is not in W.

7. **Define "dimension". Use complete sentences.**

The number of vectors in a basis for the vector space V is called the <u>dimension</u> of V.

8. **Define "eigenvalue". Use complete sentences.**

Let A be a square matrix. If there exists a non-zero vector v and a number λ with $Av = \lambda v$, then λ is called an eigenvalue of A.

9. **Prove that every 2×2 symmetric matrix has at least one real eigenvalue.**

Write $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$. The eigenvalues of A are the solutions of

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - b^2$$

$$= \lambda^2 + (-a-d)\lambda + ad - b^2.$$

The quadratic formula tells us that the solutions of the quadratic equation

$$A\lambda^2 + B\lambda + C = 0$$

are

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

In our problem, $A = 1$, $B = -a-d$ and $C = ad - b^2$. The eigenvalues of A are

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-b^2)}}{2}$$

$$= \frac{a+d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2}$$

$$= \frac{a+d \pm \sqrt{a^2 - 2ad + d^2 + 4b^2}}{2}$$

$$= \frac{a+d \pm \sqrt{(a-d)^2 + 4b^2}}{2}.$$

Notice that the number under the radical is the SUM of two perfect squares. This number is ZERO or HIGHER. So, when we take the square root, we will get real number answers (rather than complex numbers which are not real).

10. **Let v_1, v_2, and v_3 be an orthogonal set of non-zero vectors. Prove that v_1, v_2, and v_3 are linearly independent.**

Suppose $c_1v_1 + c_2v_2 + c_3v_3 = 0$. Multiply by v_1^{T}, to see that $c_1v_1^{\mathrm{T}}v_1 = 0$. I know that the number $v_1^{\mathrm{T}}v_1$ is not zero because the vector v_1 is not zero. Conclude that the number c_1 must be zero. Multiply the equation by v_2^{T} to learn that c_2 must be zero, and v_3^{T} to learn that c_3 must be zero.

11. **Consider the system of linear equations.**

$$\begin{aligned} 4x_1 \quad +ax_2 &= \quad 4 \\ ax_1 \quad +4x_2 &= \quad 4. \end{aligned}$$

(a) **Which values for a cause the system to have no solution?**
(b) **Which values for a cause the system to have exactly one solution?**
(c) **Which values for a cause the system to have an infinite number of solutions?**

Explain.

We study the system of equations $Ax = b$, where $A = \begin{bmatrix} 4 & a \\ a & 4 \end{bmatrix}$.
We know that $Ax = b$ has a unique solution for all b provided the determinant of A is non-zero. The determinant of A is $16 - a^2$; so $\det A = 0$ when $a = 4$ or -4. In other words,

if $a \neq 4$ or -4, then $Ax = b$ has a unique solution.

When $a = 4$, then the given system of equations is

$$\begin{aligned} 4x_1 \quad +4x_2 &= \quad 4 \\ 4x_1 \quad +4x_2 &= \quad 4, \end{aligned}$$

which has an infinite number of solutions. If $a = -4$, then the given system of equations is

$$\begin{aligned} 4x_1 \quad -4x_2 &= \quad 4 \\ -4x_1 \quad +4x_2 &= \quad 4, \end{aligned}$$

which has no solution. My final answer is: The sysyem of equations

has infinitely many solutions for $a = 4$ has no solution for $a = -4$ has a unique for every other choice of a.

12. **Suppose that A and B are 2×2 matrices with A non-singular. How is the null space of B related to the null space of AB, if at all? Prove your answer.**

The null space of B is equal to the null space of AB.
$\subseteq$: If x is in the null space of B, then $Bx = 0$; hence, $ABx = 0$, and x is in the null space of AB.
$\supseteq$: If x is in the null space of AB, then $ABx = 0$. The matrix A is non-singular; hence, $Bx = 0$ and x is in the null space of B.

13. **Let**

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Solve $Ax = b$. Check your answer. You might want to notice that the columns of A are an orthogonal set.

Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a solution $Ax = b$, then

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b.$$

Multiply both sides by v_i^{T} to see that $x_i = 0$ for $1 \le i \le 3$. Of course, $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not a solution of $Ax = b$. What is going on? We have learned that the only possible solution of $Ax = b$ does not work. We conclude that $\boxed{Ax = b \text{ does NOT have a solution.}}$

A different way to look at this problem is: the vectors v_1, v_2, v_3, b form an orthogonal set; hence these four vectors are linearly independent and b can not be written as a linear combination of v_1, v_2, and v_3.

14. **Find an orthogonal basis for the null space of** $A = \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix}$**. Check your answer.**

One basis for the null space of A is

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We apply the Gram-Schmidt orthogonalization process to this basis. Let $u_1 = v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

(Notice that $Au_2 = 0$ and $u_1^{\mathrm{T}} u_2 = 0$.) Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underbrace{\frac{15}{70}}_{\frac{3}{14}} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{14} \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}$$

Let

$$u_3 = \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}.$$

It is easy to check that

$$\boxed{u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}}$$

is an orthogonal basis for the null space of A.

15. **Let** $A = \begin{bmatrix} 14 & 10 \\ -5 & -1 \end{bmatrix}$**. Find a matrix** B **with** $B^2 = A$**. (I want to see the four entries in the matrix** B**.) Check your answer.**

Let

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

We see that v_1 is an eigenvector of A which belongs to $\lambda = 4$ and v_2 is an eigenvector of A which belongs to 9. In other words,

$$A \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

Let

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

We compute that

$$P^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

We have $A = PDP^{-1}$. We let

$$B = P \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}}$$

Check that

$$B^2 = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 10 \\ -5 & -1 \end{bmatrix}$$

16. **Yes or No. The vectors v_1, v_2, and v_3 are linearly independent. Are the vectors $v_1 + 2v_2 + 3v_3$, $4v_1 + 5v_2 + 6v_3$, and $7v_1 + 8v_2 + 9v_3$ also linearly independent? If yes, give a proof. If no, give an example.**

$\boxed{\text{NO}}$. The vectors $v_1 + 2v_2 + 3v_3$, $4v_1 + 5v_2 + 6v_3$, and $7v_1 + 8v_2 + 9v_3$ are linearly $\boxed{\text{DEPENDENT}}$ for every choice of v_1, v_2, and v_3 because:

$$(v_1 + 2v_2 + 3v_3) - 2(4v_1 + 5v_2 + 6v_3) + (7v_1 + 8v_2 + 9v_3) = 0$$

for EVERY choice v_1, v_2, and v_3.

17. **Let**

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 1 \\ 1 & 4 & 5 & 2 & 3 & 4 \\ 2 & 8 & 10 & 3 & 4 & 5 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}.$$

(a) Find the general solution of $Ax = b$ **.**
(b) Find three particular solutions of $Ax = b$ **.**
(c) Check that your particular solutions work.
(d) Find a basis for the column space of A **.**
(e) Find a basis for the null space of A **.**
(f) Find a basis for the row space of A **.**
(g) Express each column of A **as a linear combination of the vectors in your answer to (d).**
(h) Express each row of A **as a linear combination of the vectors in your answer to (f).** Apply Elementary row operations to

$$\left[\begin{array}{cccccc|c} 1 & 4 & 5 & 1 & 1 & 1 & 3 \\ 1 & 4 & 5 & 2 & 3 & 4 & 5 \\ 2 & 8 & 10 & 3 & 4 & 5 & 8 \end{array}\right]$$

to get

$$\left[\begin{array}{cccccc|c} 1 & 4 & 5 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) The general solution of $Ax = b$ is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 2 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

(b) Three particular solutions of $Ax = b$ are

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.}$$

(c) We check

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 1 \\ 1 & 4 & 5 & 2 & 3 & 4 \\ 2 & 8 & 10 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 1 \\ 1 & 4 & 5 & 2 & 3 & 4 \\ 2 & 8 & 10 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 1 \\ 1 & 4 & 5 & 2 & 3 & 4 \\ 2 & 8 & 10 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

(d) The vectors

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A_{*,4} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

are a basis for the column space of A.

(e) The vectors

$$\boxed{v_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

(f) The vectors

$$\boxed{\begin{array}{c} w_1 = [1 \quad 4 \quad 5 \quad 0 \quad -1 \quad -2] \\ w_2 = [0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3] \end{array}}$$

are a basis for the row space of A

(g) We see that

$$\boxed{\begin{array}{c} A_{*,2} = 4A_{*,1}, \\ A_{*,3} = 5A_{*,1}, \\ A_{*,5} = -A_{*,1} + 2A_{*,4}, \\ A_{*,6} = -2A_{*,1} + 3A_{*,4}. \end{array}}$$

(h) We see that

$$\boxed{\begin{array}{c} A_{1,*} = w_1 + w_2, \\ A_{2,*} = w_1 + 2w_2, \\ A_{3,*} = 2w_1 + 3w_2. \end{array}}$$

I use $A_{i,*}$ to mean row i of A. I use $A_{*,j}$ to mean column j of A.