Math 544 Final Exam Summer 2003

PRINT Your Name:________________________________

There are 17 problems on 10 pages. Problem 2 is worth 10 points. Problem 17 is worth 15 points. Each of the other problems is worth 5 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your course grade to you. If I don't already know your e-mail address and you want me to know it, send me an e-mail. Otherwise, get your course grade from VIP.

The solutions will be posted at my website shortly after the exam is finished.

1. Define "basis". Use complete sentences.

2. State any TWO of the four theorems about dimension. Use complete sentences.

3. Give an example of a 4×3 matrix A of rank 2. (Recall that the rank of a matrix is the dimension of its column space.)
   (a) For your matrix A, which vectors b have the property that $Ax = b$ has a solution.
   (b) For your matrix A, give an example of a non-zero vector b for which $Ax = b$ DOES have a solution.
   (c) For your matrix A, give an example of a non-zero vector b for which $Ax = b$ does NOT have a solution.

4. Let W be the set of all polynomials $f(x)$ of degree less than or equal to 3 with $f(3) = 0$. Is W a vector space? **If YES, then give a BASIS for W, no proof is needed. If NO, give an EXAMPLE which shows that W is not closed under addition or scalar multiplication.**

5. Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\\1\end{bmatrix}.$$

Find $T\left(\begin{bmatrix}5\\3\end{bmatrix}\right)$.

6. Let W be the set of 2×2 singular matrices. Is W a vector space? **If YES, then give a BASIS for W, no proof is needed. If NO, give an EXAMPLE which shows that W is not closed under addition or scalar multiplication.**

7. Define “dimension”. Use complete sentences.

8. Define “eigenvalue”. Use complete sentences.

9. Prove that every 2×2 symmetric matrix has at least one real eigenvalue.

10. Let v_1, v_2, and v_3 be an orthogonal set of non-zero vectors. Prove that v_1, v_2, and v_3 are linearly independent.

11. Consider the system of linear equations.

$$\begin{aligned} 4x_1 \quad +ax_2 &= \quad 4 \\ ax_1 \quad +4x_2 &= \quad 4. \end{aligned}$$

(a) Which values for a cause the system to have no solution?
(b) Which values for a cause the system to have exactly one solution?
(c) Which values for a cause the system to have an infinite number of solutions?

Explain.

12. Suppose that A and B are 2×2 matrices with A non-singular. How is the null space of B related to the null space of AB, if at all? Prove your answer.

13. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Solve $Ax = b$. **Check your answer.** You might want to notice that the columns of A are an orthogonal set.

14. Find an orthogonal basis for the null space of $A = \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix}$. **Check your answer.**

15. Let $A = \begin{bmatrix} 14 & 10 \\ -5 & -1 \end{bmatrix}$. Find a matrix B with $B^2 = A$. (I want to see the four entries in the matrix B.) **Check your answer.**

16. **Yes or No.** The vectors v_1, v_2, and v_3 are linearly independent. Are the vectors $v_1 + 2v_2 + 3v_3$, $4v_1 + 5v_2 + 6v_3$, and $7v_1 + 8v_2 + 9v_3$ also linearly independent? If yes, give a proof. If no, give an example.

17. Let

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 1 & 1 \\ 1 & 4 & 5 & 2 & 3 & 4 \\ 2 & 8 & 10 & 3 & 4 & 5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}.$$

(a) Find the general solution of $Ax = b$.
(b) Find three particular solutions of $Ax = b$.
(c) Check that your particular solutions work.
(d) Find a basis for the column space of A.
(e) Find a basis for the null space of A.
(f) Find a basis for the row space of A.

(g) Express each column of A as a linear combination of the vectors in your answer to (d).

(h) Express each row of A as a linear combination of the vectors in your answer to (f).