Exam 4, Summer 2003, Math 544 Solutions

PRINT Your Name:________________________________

Please also write your name on the back of the exam.

There are 9 problems on 5 pages. Problem 2 is worth 10 points. Each of the other problems is worth 5 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door later today (surely by 5:00 PM), you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Define "basis". Use complete sentences.**

The vectors $v_1, \dots, v_n$ are a basis for the vector space V if $v_1, \dots, v_n$ are linearly independent and span V.

2. **State any TWO of the four theorems about dimension. Use complete sentences.**

Theorem 1. If V is a subsapce of $\mathbb{R}^n$, then every basis for V has the same number of vectors.

Theorem 2. If V is a subsapce of $\mathbb{R}^n$, then every linearly independent subset in V is part of a basis for V.

Theorem 3. If V is a subsapce of $\mathbb{R}^n$, then every finite spanning set for V contains a basis for V.

Theorem 4. If A is a matrix, then the dimension of the column space of A plus the dimension of the null space of A is equal to the number of columns of A.

3. **Yes or No. Let v_1, v_2, v_3 be linearly independent vectors in $\mathbb{R}^n$ and let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Is it always true that $T(v_1)$, $T(v_2)$, $T(v_3)$ are linearly independent vectors? If yes, prove it. If no, give an example.**

NO! Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the function which sends every vector to zero. It is clear that T is a linear transformation but T carries the linearly independent vectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to the linearly dependent vectors $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

4. **Yes or No. Let v_1, v_2, v_3 be vectors in $\mathbb{R}^n$ and let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that $T(v_1)$, $T(v_2)$, $T(v_3)$ are linearly independent vectors in $\mathbb{R}^m$. Do the vectors v_1, v_2, v_3 have to be linearly independent? If yes, prove it. If no, give an example.**

YES! Suppose $c_1v_1 + c_2v_2 + c_3v_3 = 0$. Apply the linear transformation T and use the fact that T is a linear transformation to see that $c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = 0$. The vectors $T(v_1)$, $T(v_2)$, $T(v_3)$ are linearly independent; hence, $c_1 = c_2 = c_3 = 0$ and v_1, v_2, v_3 are linearly independent.

5. **Yes or No. Let A be a 3×4 matrix. Suppose that the column space of A has dimension 2. Does the system of equations $Ax = b$ have at least one solution for every vector b in $\mathbb{R}^3$? Explain.**

NO! The hypothesis says that the vector space of vectors in $\mathbb{R}^3$ which have the form Ax for some x in $\mathbb{R}^4$ is a two dimension object in $\mathbb{R}^3$. MOST of the vectors b in $\mathbb{R}^3$ are NOT in the column space of A. If you choose such a b, then $Ax = b$ does not have a solution.

6. **Yes or No. Let A and B be 2×2 matrices. Does**

$$\textbf{the null space of } A \subseteq \textbf{the null space of } AB$$

always happen? If yes, prove it. If no, give an example.

$\boxed{\text{NO!}}$ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We see that $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the null space of A (because $Av = 0$), but v is not in the nullspace of AB (because $ABv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$).

7. **Let** $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$. **Find a matrix B with BA equal to the identity matrix. You may do the problem any way you like, but you might want to notice that the columns of A are an orthogonal set.**

Notice that $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$. Multiply both sides by $\begin{bmatrix} 1/4 & 0 \\ 0 & 1/6 \end{bmatrix}$ to see that if $B = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/6 \end{bmatrix} A^{\mathrm{T}}$, then $BA = I$. So we take

$$\boxed{B = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 2/6 & -1/6 & -1/6 & 0 \end{bmatrix}.}$$

8. **Find the eigenvalues and the eigenvectors of the matrix** $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.

We compute

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4).$$

The eigenvalues of A are $\lambda = -1, 4$. The eigenvectors associated to $\lambda = -1$ are in the null space of $A + I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$. Elementary row operations give $\begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$. The eigenvectors of A which belong to $\lambda = -1$ are the multiples of $v = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Check that $Av = -v$. The eigenvectors associated to $\lambda = 4$ are in the null space of $A - 4I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}$. Elementary row operations give $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The eigenvectors of A which belong to $\lambda = 4$ are the multiples of $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Check that $Aw = 4w$.

9. **Find an orthogonal basis for the null space of** $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$.

One basis for the null space of A is

$$v_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

We apply Gram-Schmidt orthogonalization. Take $u_1 = v_1$. Take

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \end{bmatrix} - \frac{14}{9} \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 \\ -8 \\ -14 \\ 9 \end{bmatrix}$$

Take $u_2 = 9u_2'$. The vectors

$$u_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 1 \\ -8 \\ -14 \\ 9 \end{bmatrix}$$

are an orthogonal basis for the null space of A. Check that $Au_1 = 0$, $Au_2 = 0$, and $u_1^{\mathrm{T}} u_2 = 0$.