Exam 1, Math 544, Summer, 2003

PRINT Your Name:______________________________

Please also write your name on the back of the exam.

There are 9 problems on 6 pages. Problems 1 through 5 are worth 6 points each. Problems 6 through 9 are worth 5 points each. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door later today, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Find the GENERAL solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$\begin{array}{rrcr} 1x_1 & +2x_2 & = & 1 \\ 5x_1 & +8x_2 & = & 9 \\ 3x_1 & +5x_2 & = & 5 \end{array}$$

Start with

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 5 & 8 & 9 \\ 3 & 5 & 5 \end{array}\right]$$

Apply $R2 \mapsto R2 - 5R1$ and $R3 \mapsto R3 - 3R1$ to get

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{array}\right]$$

Apply $R1 \mapsto R1 + R2$ and $R3 \mapsto R3 - (1/2)R2$ to get

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

Apply $R2 \mapsto -(1/2)R2$ to get

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

The system of equations has a unique solution

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}}$$

This solution works because $5 - 4 = 1$, $25 - 16 = 9$, and $15 - 10 = 5$.

2. **Find the GENERAL solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$\begin{array}{rrl} 1x_1 & +2x_2 = & 1 \\ 5x_1 & +8x_2 = & 9 \\ 3x_1 & +5x_2 = & 6 \end{array}$$

Start with

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 5 & 8 & 9 \\ 3 & 5 & 6 \end{array}\right]$$

Apply $R2 \mapsto R2 - 5R1$ and $R3 \mapsto R3 - 3R1$ to get

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 3 \end{array}\right]$$

Apply $R3 \mapsto R3 - (1/2)R2$ to get

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{array}\right]$$

The bottom equation tells us that this system of equations has

$$\boxed{\text{NO SOLUTION.}}$$

3. **Find the GENERAL solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$\begin{array}{cccccccc} x_1 & +3x_2 & +\ x_3 & +9x_4 & +2x_5 & = & 7 \\ x_1 & +3x_2 & +2x_3 & +13x_4 & +x_5 & = & 3 \\ x_1 & +3x_2 & +2x_3 & +13x_4 & +2x_5 & = & 6 \end{array}$$

Start with

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 9 & 2 & 7 \\ 1 & 3 & 2 & 13 & 1 & 3 \\ 1 & 3 & 2 & 13 & 2 & 6 \end{array}\right]$$

Apply $R2 \mapsto R2 - R1$ and $R3 - R1$ to get

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 9 & 2 & 7 \\ 0 & 0 & 1 & 4 & -1 & -4 \\ 0 & 0 & 1 & 4 & 0 & -1 \end{array}\right]$$

Apply $R3 \mapsto R3 - R2$ and $R1 \mapsto R1 - R2$ to get

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 5 & 3 & 11 \\ 0 & 0 & 1 & 4 & -1 & -4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Apply $R2 \mapsto R2 + R3$ and $R1 \mapsto R1 - 3R3$ to get

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 5 & 0 & 2 \\ 0 & 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

The general solution to the system of equations is:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}.}$$

We check three specific solutions, namely when $x_2 = x_4 = 0$, when $x_2 = 1$, $x_4 = 0$, and when $x_2 = 0$, $x_4 = 1$:

$$\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 3 \end{bmatrix}.$$

We see that

$$\begin{array}{lll} 2 - 1 + 6 = 7 & -1 + 3 - 1 + 6 = 7 & -3 - 5 + 9 + 6 = 7 \\ 2 - 2 + 3 = 3 & -1 + 3 - 2 + 3 = 3 & -3 - 10 + 13 + 3 = 3 \\ 2 - 2 + 6 = 6 & -1 + 3 - 2 + 3 = 3 & -3 - 10 + 13 + 6 = 6. \end{array}$$

4. **How many solutions could a system of 4 linear equations in 3 unknowns have? List ALL of the possiblities. Explain your answer.**

A system of four linear equations in 3 unknowns could have

$$\boxed{\text{no solution, one solution, or an infinite number of solutions.}}$$

We give an example of each possibility:

$$\begin{aligned} x_1 &= 1 \\ x_1 &= 2 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

has no solution

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \\ x_1 &= 1 \end{aligned}$$

has a unique solution, and

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

has an infinite number of solutions.

5. **How many solutions could a system of 3 linear equations in 4 unknowns have? List ALL of the possiblities. Explain your answer.**

A system of 3 linear equations in 4 unknown can not possibly have a unique solution. If there are any solutions, then there must be a free variable, since the number of leading ones can be no more than the number of equations (3) and there are four variables. One of the variables which does not correspond to a leading one is free. Thus this system of equations has either

$$\boxed{\text{no solution or an infinite number of solutions.}}$$

We give an example to illustrate these possibilities.

$$\begin{aligned} x_1 &= 1 \\ x_1 &= 2 \\ x_2 + x_3 + x_4 &= 0 \end{aligned}$$

has no solution and

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$$

has an infinite number of solutions.

6. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If A, B are 2×2 symmetric matrices, then AB is a symmetric matrix.**

$\boxed{\text{FALSE}}$. The matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are both symmetric, but $AB = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, is not symmetric.

7. **True or False. (If true, explain why or give a proof. If false, give a counter example.) If A, B are 2×2 matrices, then $(A-B)(A+B) = A^2 - B^2$.**

$\boxed{\text{FALSE}}$. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We calculate that

$$(A-B)(A+B) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}.$$

On the other hand,

$$A^2 - B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We see that $(A-B)(A+B) \neq A^2 - B^2$.

8. **Give an example of 2×2 matrices A and B with $A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.**

Take $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$. It is clear that neither A nor B is equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2-2 & 4-4 \\ 4-4 & 8-8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

9. **Consider the system of linear equations.**

$$\begin{array}{rrl} 3x_1 & +ax_2 = & 3 \\ ax_1 & +3x_2 = & 3. \end{array}$$

Which values for a cause the system to have no solution? Which values for a cause the system to have exactly one solution? Which values for a cause the system to have an infinite number of solutions? Explain.

Start with

$$\left[\begin{array}{cc|c} 3 & a & 3 \\ a & 3 & 3 \end{array}\right]$$

Apply $R2 \mapsto R2 - (a/3)R1$ to get

$$\left[\begin{array}{cc|c} 3 & a & 3 \\ 0 & 3 - \frac{a^2}{3} & 3 - a \end{array}\right]$$

If $3 - \frac{a^2}{3}$ is not zero, then the system of equations has a unique solution. If $3 - \frac{a^2}{3}$ is zero and $3 - a$ is also zero, then the system of equations has infinitely many solutions. If $3 - \frac{a^2}{3}$ is zero and $3 - a$ is not zero, then the system of equations has no solution. Well, $3 - \frac{a^2}{3} = \frac{9-a^2}{3}$. So, $3 - \frac{a^2}{3}$ is zero exactly when a is 3 or -3. Of course, if $a = 3$, then $3 - a = 0$ and if $a = -3$, then $3 - a$ is not zero. So, the system of equations has

$$\boxed{\begin{cases} \text{no solution} & \text{if } a = -3 \\ \text{infinitely many solutions} & \text{if } a = 3 \\ \text{a unique solution} & \text{if } a \text{ is any number other than 3 or } -3. \end{cases}}$$