Math 544, Summer 2001, Final Exam

PRINT Your Name:________________________________

There are 14 problems on 6 pages. The exam is worth 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **NO CALCULATORS.** Your grade for the course will be available on VIP by Wednesday July 11.

1. (16 points) Let A be an $n \times n$ matrix. List 8 statements that are equivalent to the statement "A is invertible".

2. (4 points) Define "span". Use complete sentences.

3. (4 points) Define "linear combination". Use complete sentences.

4. (4 points) Define "linearly independent". Use complete sentences.

5. (4 points) Define "linear transformation". Use complete sentences.

6. (4 points) Define "one-to-one". Use complete sentences.

7. (4 points) Define "dimension". Use complete sentences.

8. (4 points) Define "column space". Use complete sentences.

9. Let $v_1, \ldots, v_m$ be vectors in $\mathbb{R}^n$. For each of the following questions, give one of the following answers: "definitely yes", "definitely no", or "sometimes". **Explain** your answer.
   (a) (3 points) Suppose $m = n$ and the vectors are linearly independent. Do the vectors span $\mathbb{R}^n$?
   (b) (3 points) Suppose $m = n + 1$. Are the vectors linearly independent?
   (c) (3 points) Suppose $m = n + 1$. Do the vectors span $\mathbb{R}^n$?
   (d) (3 points) Suppose $m = n - 1$ and the vectors are linearly independent. Do the vectors span $\mathbb{R}^n$?
   (e) (3 points) Suppose $m = n - 1$. Are the vectors linearly independent?
   (f) (3 points) Suppose $m = n - 1$. Do the vectors span $\mathbb{R}^n$?

10. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 6 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

   (a) (4 points) Find a basis for the null space of A.
   (b) (3 points) What is the dimension of the null space of A?
   (c) (4 points) Find a basis for the column space of A.
   (d) (3 points) What is the dimension of the column space of A?
   (e) (4 points) Find the general solution of $Ax = b$.
   (f) (4 points) Find the general solution of $Ax = c$.

11. (4 points) Define "eigenvalue". Use complete sentences.

12. (4 points) Diagonalize the matrix $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$.

13. (4 points) Is

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \;\middle|\; x_1 \text{ and } x_2 \text{ are real numbers} \right\}$$

a vector space? If so, explain why. If not, give an example to show that one of the rules of vector space fails to hold.

14. (4 points) Is the function $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, which is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + 3x_2 - 2x_3 \end{bmatrix},$$

a linear transformation? If so, find a matrix A with $T(v) = Av$ for all $v \in \mathbb{R}^2$. If not, give an example to show that one of the rules of linear transformation fails to hold.