SOLUTIONS to the FINAL Exam, Math 544, Spring, 2003

PRINT Your Name:______________________________

There are 20 problems on 12 pages. Each problem is worth 5 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your course grade to you. If I don't already know your e-mail address and you want me to know it, send me an e-mail. Otherwise, get your course grade from VIP.

Recall that $\mathcal{P}_n$ is the vector space of polynomials of degree at most n with real number coefficients.

Recall that the matrix A is *skew-symmetric* if $A^{\mathrm{T}} = -A$.

1. **Suppose that $T\colon \mathcal{P}_2 \to \mathcal{P}_4$ is a linear transformation, where $T(1) = x^4$, $T(x+1) = x^3-2x$, and $T(x^2+2x+1) = x$. Find $T(x^2+5x-1)$.**

Observe that $x^2+5x-1 = (x^2+2x+1) + 3(x+1) - 5(1)$; so

$$T(x^2+5x-1) = T(x^2+2x+1) + 3T(x+1) - 5T(1)$$

$$= x + 3(x^3-2x) - 5x^4 = \boxed{-5x^4+3x^3-5x}.$$

2. **Let W be the subspace of $\mathcal{P}_4$ which is defined as follows: the polynomial $p(x)$ is in W if and only if $p(1)+p(-1) = 0$ and $p(2)+p(-2) = 0$. Find the dimension of W. Explain.**

Consider the linear transformation $T\colon \mathcal{P}_4 \to \mathbb{R}^2$, which is given by $T(p(x)) = \begin{bmatrix} p(1)+p(-1) \\ p(2)+p(-2) \end{bmatrix}$. The vector space W is the null space of T. So the dimension of W is equal to the dimension of $\mathcal{P}_4$ minus the dimension of the image of T. We know that $\dim \mathcal{P}_4 = 5$, since $1, x, x^2, x^3, x^4$ is a basis for $\mathcal{P}_4$. The image of T is all of $\mathbb{R}^2$ because the image of T is a subspace of $\mathbb{R}^2$ which contains $T(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $T(x^2) = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$. The vectors $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ span $\mathbb{R}^2$. We conclude that

$$\dim W = 5 - 2 = \boxed{3}.$$

(There are many other ways to reach this answer. The most straightforward thing to do is to calculate a basis for W.)

3. **Let W be the set of 2×2 matrices whose trace is zero. Is W a vector space? If YES, then give a basis for W, no proof is needed. If NO, give an example which shows that W is not closed under addition or scalar multiplication. Recall that the *trace* of a square matrix is the sum of its diagonal elements.**

$\boxed{\text{Yes}}$, W is a vector space with basis

$$\boxed{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}.$$

4. **Let W be the set of polynomials $p(x)$ in $\mathcal{P}_3$ with $p(0) = 2$. Is W a vector space? If YES, then give a basis for W, no proof is needed. If NO, give an example which shows that W is not closed under addition or scalar multiplication.**

$\boxed{\text{No}}$, W is not a vector space. The polynomial $p(x) = 2$ is in W but the polynomial $3p(x)$, which is the constant polynomial 6, is not in W.

5. **Let W be the set of 2×2 matrices whose determinant is zero. Is W a vector space? If YES, then give a basis for W, no proof is needed. If NO, give an example which shows that W is not closed under addition or scalar multiplication.**

$\boxed{\text{No}}$, W is not a vector space. The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in W, but their sum, which is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, is not in W.

6. **Let W be the set of polynomials $p(x)$ in $\mathcal{P}_3$ with $p(2) = 0$. Is W a vector space? If YES, then give a basis for W, no proof is needed. If NO, give an example which shows that W is not closed under addition or scalar multiplication.**

$\boxed{\text{Yes}}$, W is a vector space with basis

$$\boxed{x-2, \quad (x-2)^2, \quad (x-2)^3}.$$

7. **Find $\lim_{n\to\infty} A^n$, where $A = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix}$.**

This problem would be easy if A were a diagonal matrix. Lets diagonalize A. The eigenvalues of A satisfy

$$0 = \det(A - \lambda I) = (2-\lambda)(-\frac{1}{2} - \lambda) + \frac{3}{2} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$$

$$= (\lambda - 1)(\lambda - \frac{1}{2}).$$

The eigenvalues of A are $\lambda = 1$ and $\lambda = \frac{1}{2}$. The eigenvectors which belong to $\lambda = 1$ are the nullspace of $A - I = \begin{bmatrix} 1 & \frac{3}{2} \\ -1 & -\frac{3}{2} \end{bmatrix}$. Replace row 2 by row 2 plus row 1 to get $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$. The eigenspace which belongs to $\lambda = 1$ is $x_1 = -\frac{3}{2}x_2$ and x_2 can be anything. The vector $v_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is a basis for the eigenspace which belongs to $\lambda = 1$. By the way

$$Av_1 = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -6+3 \\ 3-1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = v_1,$$

as expected. The eigenspace which belongs to $\lambda = \frac{1}{2}$ is the null space of $A - \frac{1}{2} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -1 & -1 \end{bmatrix}$. Exchange the two rows: $\begin{bmatrix} -1 & -1 \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$. Replace row 2 by row 2 plus $\frac{3}{2}$ row 1: $\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$. Multiply row 1 by -1: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The eigenspace which belongs to $\lambda = \frac{1}{2}$ is $x_1 = -x_2$ and x_2 can be anything. The vector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace which belongs to $\lambda = \frac{1}{2}$. By the way

$$Av_2 = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + \frac{3}{2} \\ 1 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$= \frac{1}{2}v_2$, as expected. Now we know that

$$A \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Let

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}.$$

We calcualate that $S^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$. We saw that $AS = SD$. It follows that $A = SDS^{-1}$ and

$$\lim_{n\to\infty} A^n = S \lim_{n\to\infty} D^n S^{-1} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 3 \\ -2 & -2 \end{bmatrix}}$$

8. **Define "linear transformation". Use complete sentences.**

The function T from the vector space V to the vector space W is a *linear transformation* if $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(rv_1) = rT(v_1)$ for all v_1 and v_2 in V and all r in $\mathbb{R}$.

9. **Define "eigenvector". Use complete sentences.**

Let A be a square matrix. The vector v is an *eigenvector* of A belonging to the eigenvalue λ provided $Av = \lambda v$ and $Aw = \lambda w$ for some non-zero vector w.

10. **Define "linearly independent". Use complete sentences.**

The vectors $v_1, \dots, v_p$ in the vector space V are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1 = \dots = c_p = 0$.

11. **Define "non-singular". Use complete sentences.** The square matrix A is *non-singular* if the only column vector x with $Ax = 0$ is $x = 0$.

12. **Define "null space". Use complete sentences.**

The *null space* of the matrix A is the set of all vectors x with $Ax = 0$.

13. **True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.) If A is a 2×2 skew-symmetric matrix, then A has at least one real eigenvalue.**

$\boxed{\text{False}}$ The eigenvectors of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are $\pm\imath$.

14. **True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.) Every 4×4 skew-symmetric matrix is singular.**

False The matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

is non-singular.

15. **True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.) If v_1, v_2, v_3 are linearly independent vectors in the vector space V and $T\colon V \to W$ is a linear transformation of vector spaces, then $T(v_1), T(v_2), T(v_3)$ are linearly independent vectors in the vector space W.**

False Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}$ which is multiplication by $[0 \quad 0 \quad 0]$. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We see that v_1, v_2, and v_3 are linearly independent in $\mathbb{R}^3$, but $T(v_1), T(v_2), T(v_3)$ are linearly dependent in $\mathbb{R}$.

16. **Find the general solution of the system of linear equations $Ax = b$. If the system of equations has more than one solution, then list three SPECIFIC solutions. CHECK that the specific solutions satisfy the equations.**

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} 6 \\ 9 \\ 10 \end{bmatrix}.$$

The same matrix A appears in problems 16, 17, and 18.

I will do the arithmetic for 16, 17, and 18 all at the same time. Consider the augmented matrix

$$\left[\begin{array}{cccc|c|c} 1 & 2 & 1 & 2 & 6 & 6 \\ 2 & 4 & 1 & 2 & 9 & 9 \\ 1 & 2 & 2 & 4 & 10 & 9 \end{array}\right]$$

Replace row 2 by row 2 minus 2 times row 1. Replace row 3 by row 3 minus row 1.

$$\left[\begin{array}{cccc|c|c} 1 & 2 & 1 & 2 & 6 & 6 \\ 0 & 0 & -1 & -2 & -3 & -3 \\ 0 & 0 & 1 & 2 & 4 & 3 \end{array}\right]$$

Replace row 1 by row 1 plus row 2. Replace row 3 by row 3 plus row 2

$$\left[\begin{array}{cccc|c|c} 1 & 2 & 0 & 0 & 3 & 3 \\ 0 & 0 & -1 & -2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Multiply row 2 by minus 1.

$$(*) \qquad \left[\begin{array}{cccc|c|c} 1 & 2 & 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Ignore the column on the far right in problem 16. The bottom row of the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

tells us that $0 = 1$. This is impossible, so the system of equations has $\boxed{\text{No Solution.}}$

17. **Find the general solution of the system of linear equations $Ax = b$. If the system of equations has**

more than one solution, then list three SPECIFIC solutions. CHECK that the specific solutions satisfy the equations.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} 6 \\ 9 \\ 9 \end{bmatrix}.$$

The same matrix A appears in problems 16, 17, and 18.

Start with the augmented matrix (*) from problem 16. Ignore column five. The resulting matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution set is

$$\boxed{\begin{array}{l} x_1 = 3 - 2x_2 \\ x_2 = \quad\quad x_2 \\ x_3 = 3 \quad\quad\quad -2x_4 \\ x_4 = \quad\quad\quad\quad\quad x_4 \end{array}}$$

Three specific solutions are:

$$\boxed{v_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.}$$

We check

$$Av_1 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 9 \end{bmatrix} = b\checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 9 \end{bmatrix} = b\checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 9 \end{bmatrix} = b\checkmark$$

18. **Find bases for the row space, column space, and null space of**

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}.$$

The same matrix A appears in problems 16, 17, and 18.

We put A into row reduced echelon form in problem 16. We may read this form from (*), simply ignore columns 5 and 6:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors

$$\boxed{\begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}}$$

are a basis for the row space of A. The vectors

$$\boxed{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}$$

are a basis for the column space of A. The vectors

$$\boxed{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

19. **Find the general solution of the system of linear equations $Ax = b$. If the system of equations has more than one solution, then list three SPECIFIC solutions. CHECK that the specific solutions satisfy the equations.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 3 & 6 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} -6 \\ -16 \\ -13 \end{bmatrix}.$$

Consider

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -6 \\ 1 & 3 & 7 & -16 \\ 1 & 3 & 6 & -13 \end{array}\right]$$

Replace row 2 with row 2 minus row 1. Replace row 3 with row 3 minus row 1:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -6 \\ 0 & 1 & 4 & -10 \\ 0 & 1 & 3 & -7 \end{array}\right]$$

Replace row 1 with row 1 minus 2 row 2. Replace row 3 with row 3 minus row 2:

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 14 \\ 0 & 1 & 4 & -10 \\ 0 & 0 & -1 & 3 \end{array}\right]$$

Replace row 2 by row 2 plus 4 row 3. Replace row 1 by row 1 minus 5 row 3:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 3 \end{array}\right]$$

Multiply row 3 by -1.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The solution is $\boxed{v = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}.}$ We check this:

$$Av = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1+4-9 \\ -1+6-21 \\ -1+6-18 \end{bmatrix} = \begin{bmatrix} -6 \\ -16 \\ -13 \end{bmatrix} = b\checkmark$$

20. **Find an orthogonal basis for the null space of** $A = \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix}$**.**

One basis for the null space of A is

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We apply the Gram-Schmidt orthogonalization process to this basis. Let $u_1 = v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

(Notice that $Au_2 = 0$ and $u_1^{\mathrm{T}} u_2 = 0$.) Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underbrace{\frac{15}{70}}_{\frac{3}{14}} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{14} \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}$$

Let

$$u_3 = \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}.$$

It is easy to check that

$$\boxed{u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}}$$

is an orthogonal basis for the null space of A.