Exam 3, Math 544, Spring, 2003, Solutions

PRINT Your Name:______________________________

Please also write your name on the back of the exam.

There are 9 problems on 6 pages. Problem 7 is worth 10 points. Each of the other problems is worth 5 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, send me an e-mail.

I will leave your exam outside my office door about 6PM today, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the exam is finished.

1. **Define "column space". Use complete sentences.** The *column space* of the matrix A is the vector space which is spanned by the columns of A.

2. **Define "null space". Use complete sentences.** The *null space* of the matrix A is the set of all column vectors v with the property that $Av = 0$.

3. **Define "basis". Use complete sentences.** A *basis* for the vector space V is a set of linearly independent vectors which span V.

4. **Solve the system of equations $Ax = b$ for**

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 4 \\ 3 \\ -2 \end{bmatrix}.$$

You may do the problem any way you like; however, you might want to notice that the columns of A form an orthogonal set.

The columns of A are an orthogonal set; hence these columns are linearly independent. It follows that $Ax = b$ has at most one solution. If the problem has a solution, then an easy way to find this solution is to multiply both sides of the equation by A^{T}. We obtain

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 8 \\ 4 \\ 6 \end{bmatrix};$$

hence,

$$\boxed{x = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}$$

Check. We see that

$$Ax = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \\ -2 \end{bmatrix} \checkmark$$

5. **Let W be the vector space which is spanned by**

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \textbf{and} \quad w_3 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find an orthogonal basis for W.

Let

$$u_1 = w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let

$$u_2 = w_2 - \frac{u_1^{\mathrm{T}} w_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Let

$$u_3 = w_3 - \frac{u_1^{\mathrm{T}} w_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} w_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Our answer is

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.}$$

Check. It is clear that u_1, u_2, and u_3 are an orthogonal set. It is also clear that

$$\begin{aligned} u_1 &= w_1 \\ u_2 &= w_2 - w_1 \\ u_3 &= w_3 - w_2 - w_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} w_1 &= u_1 \\ w_2 &= u_2 + u_1 \\ w_3 &= u_3 + u_2 + 2u_1 . \end{aligned}$$

It follows that $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$ span the same vector space.

6. **Find bases for the column space, the row space, and the null space of the matrix**

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 1 & 4 & 0 & 2 & 0 \\ 1 & 4 & 1 & 2 & 0 \\ 1 & 4 & 1 & 2 & 0 \\ 1 & 4 & 1 & 2 & 1 \end{bmatrix}.$$

Apply the following row operations:

$$\begin{aligned} R_2 &\mapsto R_2 - R_1 \\ R_3 &\mapsto R_3 - R_1 \\ R_3 &\mapsto R_4 - R_1 \\ R_4 &\mapsto R_4 - R_1 \end{aligned}$$

to get

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Apply

$$R_4 \mapsto R_4 - R_3$$

to get

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Exchange some rows to get:

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The vectors

$$\boxed{\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

are a basis for the row space of A. It is obvious to see that these vectors are linearly independent. It is not hard to see that they span the row space of A.

The vectors

$$\boxed{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix},\ \begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix},\ \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \text{ are a basis for the column space of } A.}$$

Again it is easy to see that these vectors are linearly independent and that they span the column space of A. The null space of A is the set of all vectors x with

$$\begin{aligned}x_1 &= -4x_2 - 2x_4\\ x_2 &= x_2\\ x_3 &= 0\\ x_4 &= \qquad\qquad x_4\\ x_5 &= 0\end{aligned}$$

$$\boxed{\text{A basis for the null space of } A \text{ is } \begin{bmatrix}-4\\1\\0\\0\\0\end{bmatrix},\ \begin{bmatrix}-2\\0\\0\\1\\0\end{bmatrix}}$$

It is clear that these vectors are linearly independent and that they are in the null space of A. We are very happy that $\text{rank}\, A = \text{nullityA} = 3 + 2$, which is the number of columns of A.

7. **Let A and B be an $n \times n$ matrices with A non-singular. True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.)**
   (a) **The null space of AB is equal to the null space of B.**
   (b) **The column space of AB is equal to the column space of B.**

(c) The rank of AB is equal to the rank of B.

Part (a) is true. If x is in the null space of B, then $Bx = 0$; hence, $ABx = 0$ and x is in the null space of AB. If x is in the null space of AB, then $A(Bx) = 0$; but A is non-singular, so $Bx = 0$ and x is in the null space of B.

Part (b) is false. Take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Observe that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The column space of B is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The column space of AB is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These column spaces are not equal.

Part (c) is true. The rank of AB is equal to the number of columns of AB minus the nullity of AB. The rank of B is equal to the number of columns of B minus the nullity of B. We know that AB and B have the same number of columns. Part (a) shows that B and AB have the same nullity. We conclude that B and AB have the same rank.

8. **Let a and b be vectors in $\mathbb{R}^4$, and let**

$$W = \left\{ v \in \mathbb{R}^4 \mid a^{\mathrm{T}} v = 0 \textbf{ and } b^{\mathrm{T}} v = 0 \right\}.$$

Is W a subspace of $\mathbb{R}^4$? If so, prove it. If not, give a counterexample. Any legitimate proof or counterexample will suffice.

We see that W is the null space of the matrix

$$\begin{bmatrix} a^{\mathrm{T}} \\ \hline b^{\mathrm{T}} \end{bmatrix}$$

We know that the null space of any matrix is a vector space. We conclude that

W is a vector space.

9. **Let A be a 3×4 matrix with nullity one. Does $Ax = b$ have a solution for all vectors b in $\mathbb{R}^3$? If so, prove it.**

If not, give a counterexample. Any legitimate proof or counterexample will suffice.

The dimension of the column space of A is the number of columns of A minus the dimension of the null space of A. This number is $4-1=3$. The column space of A is a three dimensional subspace of $\mathbb{R}^3$. It follows that the column space of A is equal to all of $\mathbb{R}^3$. Hence,

$$\boxed{Ax = b \text{ has a solution for all } b \text{ in } \mathbb{R}^3.}$$