Math 544 Exam 2 Solutions Spring 2003

PRINT Your Name:________________________________

Please also write your name on the back of the exam.

There are 8 problems on 4 pages. Problem 4 is worth 15 points. Each of the other problems is worth 5 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, send me an e-mail.

I will leave your exam outside my office door about 6PM today, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the exam is finished.

1. **Define "linearly independent". Use complete sentences.**

The vectors $v_1, \dots, v_r$ are *linearly independent* if the only numbers $c_1, \dots, c_r$, with $\sum_{i=1}^{r} c_i v_i = 0$, are $c_1 = \cdots = c_r = 0$.

2. **Define "non-singular". Use complete sentences.**

The $n \times n$ matrix A is *non-singular* if the only vector x in $\mathbb{R}^n$ with $Ax = 0$ is $x = 0$.

3. **Let A be an $n \times n$ matrix. List three conditions which are equivalent to the statement "A is non-singular". (I expect three new conditions in addition to "A is non-singular". Also, I do not expect you to repeat your answer to problem 2.)**
   1. The columns of A are linearly independent.
   2. The system of equations $Ax = b$ has a unique solution for every b in $\mathbb{R}^n$.
   3. The matrix A is invertible.

4. **Let** $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ **and** $b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

(a) **Find the GENERAL solution of the system of equations $Ax = b$.**
(b) **List two SPECIFIC solutions of $Ax = b$, if possible. CHECK that the specific solutions satisfy the equations.**
(c) **Are the columns of A linearly independent? Explain.**
(d) **List vectors $v_1, \dots, v_r$ so that the null space of A is the span of $v_1, \dots, v_r$. (You pick the appropriate number for r.)**
(e) **Is b in the column space of A? Explain.**

We apply Guassian Elimination to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 1 & 2 & 1 & 6 \end{array}\right].$$

Replace row 2 by row 2 minus row 1 to get

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

We learn that the general solution of $Ax = b$ is

$$\boxed{\text{The answer to (a)}: \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.}$$

Two specific solutions of $Ax = b$ are given below. In the one on the left, we took $x_2 = 0$. In the one on the right, we took $x_2 = 1$. An answer to (b) is

$$\boxed{\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.}$$

We check that these proposed solutions work. The matrix A times the left-most proposed solution is

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \checkmark .$$

The matrix A times the right-most proposed solution is

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \checkmark .$$

(c) The columns of A are linearly DEPENDENT. The short fat theorem tells us that any three vectors in $\mathbb{R}^2$ are linearly dependent.

The work we did to solve $Ax = b$ also solves $Ax = 0$. We see that the solution of $Ax = 0$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} .$$

In other words,

(d) the null space of A is the span of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

In (a) and (b), we learned that b is equal to 4 times the first column of A plus 2 times the third column of A. Thus,

(e) YES, b is in the column space of A.

5. **True or False. (If the statement is true, then PROVE the statement. If the statement is false, then give a**

COUNTEREXAMPLE.) If U and V are subspaces of $\mathbb{R}^2$, then the union $U \cup V$ is also a subspace of $\mathbb{R}^2$.

$\boxed{\text{FALSE.}}$ Here is a counter example. Let

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_2 = 0 \right\} \quad \text{and} \quad V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 = 0 \right\}.$$

The sets U and V are each subspaces of $\mathbb{R}^2$. The vectors $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are each in the union $U \cup V$ because $x \in U$ and $y \in V$, but $x + y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $U \cup V$ because $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V$.

6. **Let v_1, v_2, and v_3 be non-zero vectors in $\mathbb{R}^4$. Suppose that $v_i^{\mathrm{T}} v_j = 0$ for all subscripts i and j with $i \neq j$. Prove that v_1, v_2, and v_3 are linearly independent.**

Suppose c_1, c_2, and c_3 are numbers with

$$(*) \qquad c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$

Multiply by v_1^{T} to get

$$c_1 \cdot v_1^{\mathrm{T}} v_1 + c_2 \cdot v_1^{\mathrm{T}} v_2 + c_3 \cdot v_1^{\mathrm{T}} v_3 = 0.$$

The hypothesis tells us that $v_1^{\mathrm{T}} v_2 = 0$ and $v_1^{\mathrm{T}} v_3 = 0$. So, $c_1 \cdot v_1^{\mathrm{T}} v_1 = 0$. The hypothesis also tells us that v_1 is not zero; from which it follows that $v_1^{\mathrm{T}} v_1 \neq 0$. We conclude that $c_1 = 0$. Multiply (*) by v_2^{T} to see that $c_2 \cdot v_2^{\mathrm{T}} v_2 = 0$; hence, $c_2 = 0$, since the number $v_2^{\mathrm{T}} v_2 \neq 0$. Multiply (*) by v_3^{T} to conclude that $c_3 = 0$. We have shown that each c_i MUST be zero. We conclude that v_1, v_2, and v_3 are linearly independent.

7. **Let**

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \middle| \begin{array}{ccccc} 1x_1 & +2x_2 & +3x_3 & = & x_4 \quad \textbf{and} \\ 2x_1 & +3x_2 & +4x_3 & = & 2x_4 \end{array} \right\}.$$

Is W a vector space? Explain.

$\boxed{\text{YES, } W \text{ is a vector space.}}$ Indeed, W is the null space of the matrix

$$\begin{bmatrix} 1 & +2 & +3 & -1 \\ 2 & +3 & +4 & -2 \end{bmatrix}.$$

We proved that the null space of A is a vector space for every matrix A.

8. **Let**

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x_2 = x_1 x_3 \right\}.$$

Is W a vector space? Explain.

$\boxed{\text{NO, } W \text{ is not a vector space}}$ because W is not closed under scalar multiplication since $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in W; but $2x = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ is not in W.