Math 544, Final Exam, Solutions Fall 2009

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 200 points. There are **15** problems. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **(13 points) Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_s$ from the vector space V are *linearly independent* if the only numbers $c_1, \dots, c_s$, with $\sum\limits_{i=1}^{s} c_i v_i = 0$ are $c_1 = \cdots = c_s = 0$.

2. **(13 points) Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.**

The square matrix A is *non-singular* if the only vector v with $Av = 0$ is $v = 0$.

3. **(13 points) Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_n$ in the vector space V are a basis for V if $v_1, \dots, v_n$ span V and are linearly independent.

4. **(13 points) Define "dimension". Use complete sentences. Include everything that is necessary, but nothing more.**

The *dimension* of a vector space V is the number of vectors in a basis for V.

5. **(13 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation of vector spaces with**

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix} \quad \textbf{and} \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix}.$$

Find a matrix M with $T(v) = Mv$ for all vectors v in $\mathbb{R}^2$. Check your answer.

We see that

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix} - \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=T\left(\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}\right)=T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)-T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\4\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}.$$

We conclude that

$$\boxed{M=\begin{bmatrix}2&1\\3&1\end{bmatrix}}$$

Check.

$$M\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2+1\\3+1\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}=T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\checkmark$$

$$M\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2+2\\3+2\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}=T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\checkmark$$

6. **(13 points) Let U_1 and U_2 be subspaces of the vector space V. Does the union $U_1\cup U_2$ have to be a vector space? If yes, prove it. If no, give an example. (Recall that the vector u is in $U_1\cup U_2$ if u is in U_1 OR u is in U_2.)**

NO! Let U_1 be the subspace of $V=\mathbb{R}^2$ which is spanned by $u_1=\begin{bmatrix}1\\0\end{bmatrix}$ and U_2 be the subspace of V spanned by $u_2=\begin{bmatrix}0\\1\end{bmatrix}$. We see that u_1 and u_2 are both in $U_1\cup U_2$, but u_1+u_2 is not in $U_1\cup U_2$.

7. **(13 points) Let U_1 and U_2 be subspaces of the vector space V. Does the intersection $U_1\cap U_2$ have to be a vector space? If yes, prove it. If no, give an example. (Recall that the vector u is in $U_1\cap U_2$ if u is in U_1 AND u is in U_2.)**

YES! Let w_1 and w_2 both be in $U_1\cap U_2$ and c_1 and c_2 be real numbers. We have w_1 and w_2 both in the vector space U_1 and c_1 and c_2 are constants; thus, $c_1w_1+c_2w_2$ is in U_1. Similarly, we have w_1 and w_2 both in the vector space U_2 and c_1 and c_2 are constants; thus, $c_1w_1+c_2w_2$ is in U_2. Thus $c_1w_1+c_2w_2$ is in $U_1\cap U_2$ and $U_1\cap U_2$ is closed under both addition and scalar multiplication. Also, the zero vector is in both U_1 and U_2 so the zero vector is in $U_1\cap U_2$.

8. **(13 points) Give an example of a matrix M for which $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector belonging to the eigenvalue 1 and $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is an eigenvector belonging to the eigenvalue 2. Check your answer.**

We want

$$M \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 10 \end{bmatrix}.$$

Multiply by

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

to see that

$$M = \begin{bmatrix} 1 & 6 \\ 2 & 10 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}.}$$

Check.

$$M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7-6 \\ 10-8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ \checkmark$$

$$M \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 21-15 \\ 30-20 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix} \ \checkmark$$

9. **(13 points) Suppose that A is a matrix with distinct eigenvalues λ_1, λ_2, and λ_3. Suppose further that v_1, v_2, and v_3 are nonzero eigenvectors of A with v_i belonging to λ_i. Prove that v_1, v_2, and v_3 are linearly independent.**

Suppose c_1, c_2, c_3 are constants with

(Eq1) $$c_1v_1 + c_2v_2 + c_3v_3 = 0.$$

Multiply (Eq1) by A to see that

(Eq2) $$c_1\lambda_1v_1 + c_2\lambda_2v_2 + c_3\lambda_3v_3 = 0$$

Multiply (Eq2) by A to see that

(Eq3) $$c_1\lambda_1^2v_1 + c_2\lambda_2^2v_2 + c_3\lambda_3^2v_3 = 0$$

Consider (Eq2) minus λ_1 times (Eq1) and (Eq3) minus λ_1^2 times (Eq1):

(Eq4) $$c_2(\lambda_2 - \lambda_1)v_2 + c_3(\lambda_3 - \lambda_1)v_3 = 0$$

and

(Eq5) $$c_2(\lambda_2^2 - \lambda_1^2)v_2 + c_3(\lambda_3^2 - \lambda_1^2)v_3 = 0.$$

Observe that (Eq5) minus $(\lambda_2 + \lambda_1)$ times (Eq4) is

$$c_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)v_3 = 0.$$

The vector v_3 is not zero. The constant $(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$ is not zero. So the constant c_3 must be zero. Now look at (Eq4): $c_2(\lambda_2 - \lambda_1)v_2 = 0$. The vector $(\lambda_2 - \lambda_1)v_2$ is not zero; so c_2 must be zero. Now look at (Eq1): $c_1 v_1 = 0$. The vector v_1 is not zero; so $c_1 = 0$. We have shown that the only way for (Eq1) to happen is with all c_i equal to zero. We conclude that v_1, v_2, v_3 are linearly independent.

10. **(13 points) Suppose that $V \subseteq W$ are vector spaces and w_1, w_2, w_3 is a basis for W. Suppose further that w_1 and w_2 are in V, but w_3 is not in V. Do you have enough information to know the exact value of $\dim V$? If yes, prove it. If no, then give enough examples to show that $\dim V$ has not yet been determined.**

$\boxed{\text{We know that } \dim V = 2}$. Indeed, w_1 and w_2 are linearly independent vectors in V; so w_1 and w_2 is the beginning of a basis for V and $\dim V \geq 2$. The only three dimensional subspace of W is all of W. Thus, $\dim V \leq 2$, and indeed, $\dim V = 2$.

11. **(13 points) Suppose that $V \subseteq W$ are vector spaces and w_1, w_2, w_3, w_4 is a basis for W. Suppose further that w_1 and w_2 are in V, but neither w_3 nor w_4 is not in V. Do you have enough information to know the exact value of $\dim V$? If yes, prove it. If no, then give enough examples to show that $\dim V$ has not yet been determined.**

NO! Let $W = \mathbb{R}^4$ and

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- In our first example we take V to be spanned by w_1 and w_2. In this case, $\dim V = 2$.
- In our second example we take V to be spanned by w_1, w_2, and $w_3 + w_4$. In this case, $\dim V = 3$ and neither w_3 nor w_4 is in V!

12. **(13 points) Recall that $\mathcal{P}_4$ is the vector space of polynomials of degree at most 4. Let W be the following subspace of $\mathcal{P}_4$:**

$$W = \{p(x) \in \mathcal{P}_4 \mid p(1) + p(-1) = 0 \quad \textbf{and} \quad p(2) + p(-2) = 0\}.$$

Find a basis for W.

Every element of W has the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

where

$$\begin{cases} p(1) + p(-1) = 0 \\ p(2) + p(-2) = 0 \end{cases}$$

In other words,

$$\begin{cases} (a_0 + a_1 + a_2 + a_3 + a_4) + (a_0 - a_1 + a_2 - a_3 + a_4) = 0 \\ (a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4) + (a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4) = 0 \end{cases}$$

In other words,

$$\begin{cases} 2a_0 + 2a_2 + 2a_4 = 0 \\ 2a_0 + 8a_2 + 32a_4 = 0 \end{cases}$$

In other words,

$$\begin{cases} a_0 + a_2 + a_4 = 0 \\ a_0 + 4a_2 + 16a_4 = 0 \end{cases}$$

Subtract Eq1 from Eq2 to get:

$$\begin{cases} a_0 + a_2 + a_4 = 0 \\ \qquad 3a_2 + 15a_4 = 0 \end{cases}$$

$$\begin{cases} a_0 + a_2 + a_4 = 0 \\ \qquad a_2 + 5a_4 = 0 \end{cases}$$

Subtract equation 2 from Eq1:

$$\begin{cases} a_0 - 4a_4 = 0 \\ \qquad a_2 + 5a_4 = 0 \end{cases}$$

So a_1, a_3, a_4 are free variables and the value of a_0 and a_2 is determined by the value of the free variables: $a_0 = 4a_4$ and $a_2 = -5a_4$. So every element of W has the form $a_1 x + a_3 x^3 + a_4(4 - 5x^2 + x^4)$. The polynomials $\boxed{x, x^3, 4 - 5x^2 + x^4}$ span W and are linearly independent; they form a basis for W. By the way, $4 - 5x^2 + x^4$ vanishes at $1, -1, 2, -2$.

13. **(13 points) Find an orthogonal basis for the null space of** $A = \begin{bmatrix} 1 & 3 & 4 & 5 \end{bmatrix}$**. Check your answer.**

Let V be the null space of A. We start with the basis

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

for V. Let $u_1 = v_1$. Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{12}{10} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ 6 \\ -5 \\ 0 \end{bmatrix}.$$

Let $u_2 = 5u_2' = \begin{bmatrix} 2 \\ 6 \\ -5 \\ 0 \end{bmatrix}$. Be sure to check that $u_2 \in V$ and u_1 and u_2 are orthogonal. Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{15}{10} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{10}{65} \begin{bmatrix} 2 \\ 6 \\ -5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{13} \begin{bmatrix} 2 \\ 6 \\ -5 \\ 0 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 5 \\ 15 \\ 20 \\ -26 \end{bmatrix}.$$

Let $u_3 = 26u_3' = \begin{bmatrix} 5 \\ 15 \\ 20 \\ -26 \end{bmatrix}$. Be sure to check that $u_3 \in V$ and u_3 is orthogonal to u_1 and u_2 are orthogonal. We conclude that an orthogonal basis for the null space of A is

$$\boxed{u_1 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 6 \\ -5 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 5 \\ 15 \\ 20 \\ -26 \end{bmatrix}.}$$

14. **(13 points) Let** $A = \begin{bmatrix} 15 & -7 \\ 14 & -6 \end{bmatrix}$**. Find a matrix** B **with** $B^3 = A$**. Check your answer.**

We compute

$$\det A - \lambda I = (15-\lambda)(-6-\lambda) + 98 = \lambda^2 - 9\lambda + 8 = (\lambda - 1)(\lambda - 8).$$

The eigenvectors that belong to $\lambda = 1$ are in the null space of $\begin{bmatrix} 14 & -7 \\ 14 & -7 \end{bmatrix}$. One such vector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The eigenvectors that belong to $\lambda = 8$ are in the null space of $\begin{bmatrix} 7 & -7 \\ 14 & -14 \end{bmatrix}$. One such vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, we have

$$A \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}.$$

We notice that the inverse of $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$. We take

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}}$$

Check. We compute

$$B^3 = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & -7 \\ 14 & -6 \end{bmatrix} \checkmark$$

15. **(18 points) Check your answers. Let** A **be the matrix**

$$A = \begin{bmatrix} 1 & 4 & 1 & 5 & 1 & 13 \\ 1 & 4 & 2 & 5 & 2 & 20 \\ 2 & 8 & 3 & 10 & 3 & 33 \end{bmatrix}$$

The row operations $R2 - R1$ and $R3 - 2R1$ yield:

$$\begin{bmatrix} 1 & 4 & 1 & 5 & 1 & 13 \\ 0 & 0 & 1 & 0 & 1 & 7 \\ 0 & 0 & 1 & 0 & 1 & 7 \end{bmatrix}$$

The row operations $R1 - R2$ and $R3 - R2$ yield:

$$\begin{bmatrix} 1 & 4 & 0 & 5 & 0 & 6 \\ 0 & 0 & 1 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) **Find a basis for the null space of A.**

The null space of A is the set of vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_6 \begin{bmatrix} -6 \\ 0 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where x_2, x_4, x_5, and x_6 are arbitrary. Thus, the vectors

$$\boxed{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -6 \\ 0 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

(b) **Find a basis for the column space of A.**

In the reduced matrix, the leading ones appear in columns 1 and 3; so a basis for the column space of A is columns 1 and 3 from A:

$$\boxed{v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.}$$

(c) **Find a basis for the row space of A.**

The non-zero rows of the reduced matrix form a basis for the row space of A:

$$\boxed{u_1 = [1 \quad 4 \quad 0 \quad 5 \quad 0 \quad 6] \text{ and } u_2 = [0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 7].}$$

(d) **Write each column of A as a linear combination of your answer to (b).**

Let C_i be the i^{th} column of A. Observe that

$$\boxed{C_1 = v_1, \quad C_2 = 4v_1, \quad C_3 = v_2, \quad C_4 = 5v_1, \quad C_5 = v_2, \quad \text{and} \quad C_6 = 6v_1 + 7v_2.}$$

(e) **Write each row of A as a linear combination of your answer to (c).**

Let R_i be the i^{th} row of A. Observe that

$$\boxed{R_1 = 1u_1 + 1u_2, \quad R_2 = 1u_1 + 2u_2, \quad \text{and} \quad R_3 = 2u_2 + 3u_2.}$$