**Math 544, Final Exam, Fall 2006**

Write your answers as legibly as you can on the blank sheets of paper provided.

**Please leave room in the upper left corner for the staple.**

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 11 problems. The exam is worth a total of 100 points.

SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as I have graded the exam. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **(7 points) Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_r$ in the vector space $V$ are *linearly independent* if the only numbers $c_1, \dots, c_r$, with $\sum_{i=1}^{r} c_i v_i = 0$, are $c_1 = \cdots = c_r = 0$.

2. **(7 points) Define "span". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, v_2, \dots, v_n$ in the vector space $V$ *span* $V$ if every vector in $V$ is equal to a linear combination of $v_1, v_2, \dots, v_n$.

3. **(7 points) Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.**

A *basis* for the vector space $V$ is a set of vectors in $V$ which span $V$ and are linearly independent.

4. **(12 points) Let** $A = \begin{bmatrix} 3 & 5 \\ -1 & -\frac{3}{2} \end{bmatrix}$**. Find** $\lim_{n\to\infty} A^n$**.**

You must compute the eigenvalues $1$ and $\frac{1}{2}$ for $A$. Observe that $v_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ is an eigenvector of $A$ which belongs to $\lambda = 1$ because

$$Av_1 = \begin{bmatrix} 3 & 5 \\ -1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 15-10 \\ -5+3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = v_1.$$

Observe that $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ which belongs to $\lambda = \frac{1}{2}$ because

$$Av_2 = \begin{bmatrix} 3 & 5 \\ -1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+5 \\ 2-\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} = \tfrac{1}{2} v_2.$$

Thus, we know $AS = SD$, where

$$S = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

It is easy to see that

$$S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

and $A = SDS^{-1}$. We see that

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} (SDS^{-1})^n = S(\lim_{n\to\infty} D^n)S^{-1} = S(\lim_{n\to\infty} \begin{bmatrix} 1^n & 0 \\ 0 & (1/2)^n \end{bmatrix})S^{-1}$$

$$= S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -2 & -4 \end{bmatrix}$$

5. **(13 points) Let $W$ be the subspace of $\mathbb{R}^4$ which is spanned by**

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \textbf{and} \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

**Find an orthogonal basis for $W$.**

Let $u_1 = w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Let

$$u_2' = w_2 - \frac{u_1^{\mathrm{T}} w_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Let $u_2 = 4u_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Let

$$u_3' = w_3 - \frac{u_1^{\mathrm{T}} w_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} w_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Let $u_3 = 3u_3' = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$. Our orthogonal basis for $W$ is

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}}$$

6. **(13 points) Let**

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}, \quad \textbf{and} \quad b = \begin{bmatrix} -2 \\ 20 \\ 10 \\ 33 \end{bmatrix}.$$

(a) **Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.**

**We apply row operations to the augmented matrix**

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & -2 \\ 3 & 12 & 1 & 5 & 5 & 20 \\ 2 & 8 & 1 & 3 & 2 & 10 \\ 5 & 20 & 2 & 8 & 8 & 33 \end{bmatrix}$$

**Replace row 2 by row 2 minus 3 row 1.**
**Replace row 3 by row 3 minus 2 row 1.**
**Replace row 4 by row 4 minus 5 row 1 to get**

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 8 & 26 \\ 0 & 0 & 1 & -1 & 4 & 14 \\ 0 & 0 & 2 & -2 & 13 & 43 \end{bmatrix}$$

**Replace row 3 by row 3 minus row 2.**
**Replace row 4 by row 4 minus 2 row 2.**

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 8 & 26 \\ 0 & 0 & 0 & 0 & -4 & -12 \\ 0 & 0 & 0 & 0 & -3 & -9 \end{bmatrix}$$

**Divide row 3 by** $-4$**. Divide row 4 by** $-3$**.**

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 8 & 26 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

**Replace row 2 by row 2 minus 8 times row 3. Replace row 1 by row 1 plus row 3. Replace row 4 by row 4 minus row 3.**

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**The general solution of the system of equations is**

$$\boxed{\begin{array}{lccc} x_1 = & 1 & -4x_2 & -2x_4 \\ x_2 = & & x_2 & \\ x_3 = & 2 & & +x_4 \\ x_4 = & & & x_4 \\ x_5 = & 3 & & \end{array}}$$

**Three specific solutions are**

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 3 \end{bmatrix}.}$$

**(b) Find a basis for the null space of** $A$**.**

$$\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

**(c) Find a basis for the column space of $A$.**

$$A_{*,1} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \quad A_{*,3} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad A_{*,5} = \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix}$$

**(d) Find a basis for the row space of $A$.**

$$\begin{aligned} w_1 &= [1 \quad 4 \quad 0 \quad 2 \quad 0] \\ w_2 &= [0 \quad 0 \quad 1 \quad -1 \quad 0] \\ w_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 1] \end{aligned}$$

**(e) Express each column of $A$ in terms of your answer to (c).**

$$A_{*,2} = 4A_{*,1}, \quad A_{*,4} = 2A_{*,1} - A*,3.$$

**(f) Express each row of $A$ in terms of your answer to (d).**

$$A_{1,*} = w_1 - w_3, \quad A_{2,0} = 3w_1 + w_2 + 5w_3,$$
$$A_{3,*} = 2w_1 + w_2 + 2w_3, \quad A_{4,*} = 5w_1 + 2w_2 + 8w_3.$$

7. **(7 points) Let $T\colon V \to W$ be a linear transformation of finite dimensional vector spaces. Let $w_1, \ldots, w_r$ in $W$ be a basis for the image of $T$. Let $v_1, \ldots, v_r$ be vectors in $V$ with $T(v_i) = w_i$, for $1 \le i \le r$. Let $u_1, \ldots, u_s$ in $V$ be a basis for the null space of $T$. Prove that $v_1, \ldots, v_r, u_1, \ldots, u_s$ is a basis for $V$.**

We first show linear independence. Suppose that $a_i$ and $b_j$ are numbers with

$$(*) \qquad \sum_{i=1}^{r} a_i v_i + \sum_{j=1}^{s} b_j u_j = 0.$$

Apply the linear transformation $T$ to see that

$$\sum_{i=1}^{r} a_i w_i = 0.$$

The vectors $w_1, \dots, w_r$ are linearly independent, so each $a_i$ must be zero, Now (*) tells us that

$$\sum_{j=1}^{s} b_j u_j = 0.$$

The vectors $u_1, \dots, u_s$ are linearly independent so each $b_j$ must also be zero.

Now we show that $v_1, \dots, v_r, u_1, \dots, u_s$ span $V$. Take $v \in V$. The vector $T(v)$ is in the image of $T$, so there are numbers $a_1, \dots, a_r$ with $T(v) = \sum\limits_{i=1}^{r} a_i w_i$. Of course $w_i = T(v_i)$. It follows that $v - \sum\limits_{i=1}^{r} a_i v_i$ is in the null space of $T$ that is, $v - \sum\limits_{i=1}^{r} a_i v_i$ may be written as a linear combination of $u_1, \dots, u_s$. In other words, $v$ may be written as a linear combination of $v_1, \dots, v_r, u_1, \dots, u_s$, and the proof is complete.

8. **(7 points) Let $V$ be a vector space and let $T\colon V \to V$ be a linear transformation. Suppose that $v_1, v_2, v_3$ are non-zero vectors in $V$ which are eigenvectors which belong to three distinct eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$, respectively. Prove that $v_1, v_2, v_3$ are linearly independent.**

We first show that $v_1$ and $v_2$ are linearly independent. Suppose $a_1$ and $a_2$ are numbers with

$$(**) \qquad a_1 v_1 + a_2 v_2 = 0.$$

Multiply (**) by $\lambda_1$ to get

$$a_1 \lambda_1 v_1 + a_2 \lambda_1 v_2 = 0.$$

Multiply (**) by $A$ to get

$$a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0.$$

Subtract to learn that $a_2(\lambda_2 - \lambda_1)v_2 = 0$. The vector $v_2$ is not zero. The number $\lambda_2 - \lambda_1$ is not zero; so, the number $a_2$ must be zero. Look back at (**) to see that $a_1$ must also be zero.

Now we deal with all three vectors. Suppose $a_1$, $a_2$, and $a_3$ are numbers with

(***) $$a_1v_1 + a_2v_2 + a_3v_3 = 0$$

Multiply (***) by $\lambda_3$ to see

$$a_1\lambda_3v_1 + a_2\lambda_3v_2 + a_3\lambda_3v_3 = 0$$

Multiply (***) by $A$ to see

$$a_1\lambda_1v_1 + a_2\lambda_2v_2 + a_3\lambda_3v_3 = 0$$

Subtract to learn that

$$a_1(\lambda_1 - \lambda_3)v_1 + a_2(\lambda_2 - \lambda_3)v_2 = 0.$$

We already showed that $v_1$ and $v_2$ are linearly independent; so, $a_1(\lambda_1 - \lambda_3) = 0$ and $a_2(\lambda_2 - \lambda_3) = 0$. The numbers $\lambda_2 - \lambda_3$ and $\lambda_1 - \lambda_3$ are not zero; hence $a_1$ and $a_2$ must be zero. Look back at (***) to see that $a_3$ must also be zero.

9. **(7 points) Is the the determinant function from the vector space of $2 \times 2$ matrices to $\mathbb{R}$ a linear transformation? Explain thoroughly.**

NO. The determinant of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ plus the determinant of $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is zero plus zero which is zero. On the other hand if I add the matrices first and then take the determinant I get the determinant of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is $1$.

10. **(7 points) Recall that $\mathbb{R}[x]$ is the vector space of polynomials in one variable $x$ with real number coefficients. Consider the function $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$, which is given by $T(f) = x^2f$ for each polynomial $f \in \mathbb{R}[x]$. Is $T$ a linear transformation? Explain thoroughly.**

Yes. If $f$ and $g$ are polynomials, then

$$T(f+g) = x^2(f+g) = x^2f + x^2g = T(f + T(g).$$

If $c$ is a number, then

$$T(cf) = x^2cf = cx^2f = cT(f).$$

11. **(13 points) In this problem, if $M$ is a matrix, then let $\mathcal{I}(M)$ be the Column space of $M$. Let $A$ and $B$ be $n \times n$ matrices. For each question: if the answer is yes, then prove the statement; if the answer is no, then give a counter example.**

(a) **Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?**

No. Take $B$ to be the identity matrix and $A$ to be the zero matrix.

(b) **Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?**

No. Take $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ We see that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the column space of $AB$ and not in the column space of $B$.

(c) **Suppose $B$ is non-singular. Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?**

No. Take $B$ to be the identity matrix and $A$ to be the zero matrix.

(d) **Suppose $B$ is non-singular. Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?**

Yes. In this case the column space of $B$ is all of $\mathbb{R}^n$.

(e) **Suppose $A$ is non-singular. Does $\mathcal{I}(B)$ have to be a subset of $\mathcal{I}(AB)$?**

No. Take $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ We see that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the column space of $B$ and not in the column space of $AB$.

(f) **Suppose $A$ is non-singular. Does $\mathcal{I}(AB)$ have to be a subset of $\mathcal{I}(B)$?**

No. Take $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ We see that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the column space of $AB$ and not in the column space of $B$.