Math 544, Exam 3, Fall 2006

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 50 points.

No Calculators or Cell phones.

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as I have graded the exam. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **(5 points) Define "closed under addition". Use complete sentences. Include everything that is necessary, but nothing more.**

The subset V of $\mathbb{R}^n$ is closed under addition if whenever v_1 and v_2 are elements of V, then $v_1 + v_2$ is also an element of V.

2. **(5 points) Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.**

A basis for a vector space V is a linearly independent subset of V which spans V.

3. **(5 points) Let A be an $n \times n$ matrix and let $W = \{v \in \mathbb{R}^n | \, Av = 2v\}$. Is W a subspace of $\mathbb{R}^n$? If yes, then give a complete, correct, proof. If no, then give an explicit example that shows that W is not a subspace of of $\mathbb{R}^n$.**

YES. The set W is equal to the nullspace of the matrix $A - 2I$. The null space of every matrix is a vector space.

4. **(5 points) Let U and V be subspaces of $\mathbb{R}^n$. Is the union $U \cup V$ a subspace of $\mathbb{R}^n$? If yes, then give a complete, correct, proof. If no, then give an explicit example that shows that $U \cup V$ is not a subspace of $\mathbb{R}^n$.**

NO. Let $U = \{ \begin{bmatrix} a \\ 0 \end{bmatrix} \mid a \in \mathbb{R} \}$ and $V = \{ \begin{bmatrix} 0 \\ b \end{bmatrix} \mid b \in \mathbb{R} \}$. We see that $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both elements of $U \cup V$, but $u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $U \cup V$.

5. **(10 points) State the four theorems about dimension.**

Theorem 1. If V is a subsapce of $\mathbb{R}^n$, then every basis for V has the same number of vectors.

Theorem 2. If V is a subsapce of $\mathbb{R}^n$, then every linearly independent subset in V is part of a basis for V.

Theorem 3. If V is a subsapce of $\mathbb{R}^n$, then every finite spanning set for V contains a basis for V.

Theorem 4. If A is a matrix, then the dimension of the column space of A plus the dimension of the null space of A is equal to the number of columns of A.

6. (10 points) **Let V and W be subspaces of $\mathbb{R}^n$ with $V \subseteq W$.**
   (a) **Does the dimension of V have to be $\leq$ the dimension of W? If yes, then give a complete, correct, proof. If no, then give an explicit example.**

YES. A basis for V is a linearly independent set in W. Every linearly indepndent set in W is conatained in a basis for W. It follows that the dimension of V, which is the number of vectors in a basis for V, is less than or equal to the dimension of W, which is the number of vectors in a basis for W.

   (b) **Suppose $\dim V = \dim W$. Does V have to equal W? If yes, then give a complete, correct, proof. If no, then give an explicit example.**

YES. Let $v_1, \ldots, v_p$ be a basis for V. Part (a) shows that $v_1, \ldots, v_p$ is part of a basis for W. However every basis for W has p vectors. So $v_1, \ldots, v_p$ are already a basis for W. In particular $v_1, \ldots, v_p$ span W. Every element in W is automatically also in V. The sets V and W are equal.

7. (10 points) **Let A and B be $n \times n$ matrices, with A non-singular. Answer each question. If the answer is yes, then give a complete, correct, proof. If the answer is no, then give an example.**

   (a) **Does the null space of B have to be equal the null space of AB?**

YES. If v is in the null space of B, then $bv = 0$, so $ABv = 0$ and v is in the null space of AB.
The other direction is trickier. The hypothesis that A is non-singular guarantees that A has an inverse. If v is in the null space of AB, then $ABv = 0$. Multiply by A^{-1} to see that $Bv = 0$ and v is in the null space of B.

   (b) **Does the dimension of the null space of B have to equal the dimension of the null space of AB?**

YES. The two spaces are EQUAL. Of course they have the same dimension.

   (c) **Does the column space of B have to equal the column space of AB?**

NO. Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We see that A is non-singular and that $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The column space of B is the x-axis. The column space of AB is the y-axis. Thus, AB and B have different column spaces.

   (d) **Does the dimension of the column space of B have to equal the dimension of the column space of AB?**

YES. Use Theorem 4 and part (b):

$$\operatorname{rank} B = n - \operatorname{nullity} B = n - \operatorname{nullity} AB = \operatorname{rank} AB.$$