Math 544, Exam 2, Fall 2006

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 50 points. There are 9 problems. Problem 9 is worth 10 points. Every other problem is worth 5 points.

No Calculators or Cell phones.

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as I have graded the exam. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **Define "null space". Use complete sentences. Include everything that is necessary, but nothing more.**

The *null space* of the matrix A is the set of all vectors v such that $Av = 0$.

2. **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^m$ are *linearly independent* if the ONLY numbers $c_1, \dots, c_p$, with $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

3. **Define "subspace of $\mathbb{R}^n$". Use complete sentences. Include everything that is necessary, but nothing more.**

The subset V of $\mathbb{R}^n$ is a *subspace* of $\mathbb{R}^n$ if $0 \in V$; V is closed under addition (that is, if v_1 and v_2 are elements of V, then $v_1 + v_2$ is an element of V); and V is closed under scalar multiplication (that is, if v is in V and $c \in \mathbb{R}$, then $cv \in V$.)

4. **Let** $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, |x_1| = |x_2| \right\}$ **. Is** W **a subspace of** $\mathbb{R}^2$ **? If yes, then give a complete, correct, proof. If no, then give an explicit example that shows that** W **is not a subspace of of** $\mathbb{R}^2$ **.**

NO! The vecotrs $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are in V, but the sum $v_1 + v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not in V.

5. **Let** $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, \begin{matrix} x_1 + 2x_2 = 3x_3 \\ 2x_1 - 3x_2 = 4x_4 \end{matrix} \right\}$ **. Is** W **a subspace of** $\mathbb{R}^4$ **? If yes, then give a complete, correct, proof. If no, then give an explicit example that shows that** W **is not a subspace of of** $\mathbb{R}^4$ **.**

YES! The set W is the null space of the matrix $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -3 & 0 & -4 \end{bmatrix}$. We proved that the null space of every matrix is a vector space.

6. **Let** v_1, v_2, v_3, v_4 **be vectors in** $\mathbb{R}^5$ **, with** v_1, v_2, v_3 **linearly independent and** $v_4 = 2v_1 + 3v_2 + 4v_3$ **. Do the vectors** v_2, v_3, v_4 **have to be linearly independent? If yes, then give a complete, correct, proof. If no, then give an example.**

YES! Suppose

(**) $$c_2v_2 + c_3v_3 + c_4v_4 = 0.$$

It follows that $c_2v_2 + c_3v_3 + c_4(2v_1 + 3v_2 + 4v_3) = 0$; so,

(*) $$2c_4v_1 + (c_2 + 3c_4)v_2 + (c_3 + 4c_4)v_3 = 0.$$

The hypothesis tells us that the vectors v_1, v_2, v_3 are linearly independent. It follows that the coefficients of (*) MUST be zero. In other words, $2c_4 = 0$, $c_2 + 3c_4 = 0$, and $c_3 + 4c_4 = 0$. Thus, c_4 must be 0 (from the first equation), and c_2 must be 0 from the second equation, and c_3 must be 0 from the third equation. We have shown that the ONLY way for (**) to occur is for $c_2 = c_3 = c_4 = 0$. It follows that v_2, v_3, v_4 are linearly independent vectors.

7. **Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^5$, with v_1, v_2, v_3 linearly independent and v_4 equal to a linear combination of v_1, v_2, and v_3. Do the vectors v_2, v_3, v_4 have to be linearly independent? If yes, then give a complete, correct, proof. If no, then give an example.**

NO! Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

We see that v_1, v_2, v_3 are linearly independent (since if $c_1v_1 + c_2v_2 + c_3v_3 = 0$, then

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and c_1, c_2, and c_3 must all be zero); v_4 is a linear combination of v_1, v_2, and v_3 (since $v_4 = 0v_1 + 0v_2 + 2v_3$); but v_2, v_3, v_4 are linbearly dependent because $2v_3 - v_4 = 0$.

8. **How many solutions does a system of 4 linear equations in 3 unknowns have? Justify your answer very thoroughly.**

A system of 4 linear equations in 3 unknowns might have no solutions (here is an example $x_1 = 1$, $x_1 = 2$, $x_2 = 0$, and $x_3 = 0$) or it might have a unique solution (here is an example: $x_1 = 1$, $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$) or it might have an infinite number of solitions (here is an example: $x_1 = 1$, $x_2 + x_3 = 0$, $x_2 + x_3 = 0$, $x_2 + x_3 = 0$). The hypothesis of this problem did not rule out anything.

9. **Let A and B be $n \times n$ matrices. Answer each question. If the answer is yes, then give a complete, correct, proof. If the answer is no, then give an example.**
   (a) **Does the null space of A have to be a subset of the null space of AB?**

NO! Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We see that $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the null space of A (since $Av = 0$), but v is not in the null space of $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (since $ABv \neq 0$).

(b) **Does the null space of B have to be a subset of the null space of AB?**

YES! Take v from the null space of B. Observe that $ABv = A(Bv) =^{\dagger} A0 = 0$. Thus, v must be in the null space of AB. (The equality at † holds because v is in the null space of B.)

(c) **Does the null space of AB have to be a subset of the null space of A?**

NO! Take A to be the identity matrix and B to be the zero matrix. We see that AB is the zero matrix. Thus, all of $\mathbb{R}^n$ is in the null space of AB; but only zero is in the null space of A.

(d) **Does the null space of AB have to be a subset of the null space of B?**

NO! Take A to be the zero matrix and B to be the identity matrix. We see that AB is the zero matrix. Thus, all of $\mathbb{R}^n$ is in the null space of AB; but only zero is in the null space of B.