Math 544, Final Exam, Fall 2005, Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 16 problems. Problem 1 is worth 10 points. Each of the other problems is worth 6 points. The exam is worth a total of 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

I WILL GRADE YOUR EXAM ON FRIDAY. Once your exam is graded, I will send your grade to VIP. If the grade isn't on VIP, then I also do not know it. If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website shortly after the exam is finished.

1. **Let**

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \textbf{and} \quad c = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

(a) **Find the general solution of $Ax = b$. List three specific solutions, if possible. Check your solutions.**
(b) **Find the general solution of $Ax = c$. List three specific solutions, if possible. Check your solutions.**
(c) **Find a basis for the null space of A.**
(d) **Find a basis for the column space of A.**
(e) **Find a basis for the row space of A.**
(f) **Express each column of A in terms of your answer to (d).**
(g) **Express each row of A in terms of your answer to (e).**

We study the augmented matrix

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 2 & 4 & 6 & 2 & 1 & 5 & 2 & 2 \\ 2 & 4 & 6 & 1 & 2 & 5 & 2 & 2 \\ 2 & 4 & 6 & 1 & 1 & 4 & 2 & 3 \end{array}\right].$$

Apply $R_2 \mapsto R_2 - 2R_1$, $R_3 \mapsto R_3 - 2R_1$, and $R_4 \mapsto R_4 - 2R_1$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right].$$

Exchange rows 2 and 3 to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 + R_2$ and $R_4 \mapsto R_4 - R_2$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 + R_3$ and $R_4 \mapsto R_4 - R_3$ to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Multiply rows 2 and 3 by -1 to obtain

$$\left[\begin{array}{cccccc|cc} 1 & 2 & 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

The general solution to $Ax = b$ is

(a) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{for any } x_2,\ x_3,\ x_6 \text{ in } \mathbb{R}.$$

Four specific solutions are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

(I obtained v_1 by setting $x_2 = x_3 = x_6 = 0$; v_2 by setting $x_2 = 1, x_3 = x_6 = 0$; v_3 by setting $x_3 = 1, x_2 = x_6 = 0$; and v_4 by setting $x_6 = 1, x_2 = x_3 = 0$.) I check that

$$Av_1 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b; \checkmark$$

$$Av_4 = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 6 & 2 & 1 & 5 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 2 & 4 & 6 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = b. \checkmark$$

(b) The equations $Ax = c$ have $\boxed{\text{NO solution.}}$

(c) The vectors

$$\boxed{w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$$

are a basis for the null space of A.

(d) The vectors

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad A_{*,4} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad A_{*,5} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}}$$

are a basis for the column space of A .

(e) The vectors

$$\boxed{\begin{aligned} z_1 &= \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \\ z_2 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \\ z_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}}$$

are a basis for the row space of A .

(f)

$$\boxed{A_{*,2} = 2A_{*,1}, \quad A_{*,3} = 3A_{*,1}, \quad A_{*,6} = A_{*,1} + A_{*,4} + A_{*,5}.}$$

(g)

$$\boxed{\begin{aligned} A_{1,*} &= z_1 + z_2 + z_3, \\ A_{2,*} &= 2z_1 + 2z_2 + z_3, \\ A_{3,*} &= 2z_1 + z_2 + 2z_3, \\ A_{4,*} &= 2z_1 + z_2 + z_3. \end{aligned}}$$

2. **State any two of the four dimension Theorems.**

Theorem 1. If V is a subsapce of $\mathbb{R}^n$, then every basis for V has the same number of vectors.

Theorem 2. If V is a subsapce of $\mathbb{R}^n$, then every linearly independent subset in V is part of a basis for V .

Theorem 3. If V is a subsapce of $\mathbb{R}^n$, then every finite spanning set for V contains a basis for V .

Theorem 4. If A is a matrix, then the dimension of the column space of A plus the dimension of the null space of A is equal to the number of columns of A .

3. **Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.**

A basis for a vector space V is a linearly independent subset of V which spans V .

4. **Define "linear transformation". Use complete sentences. Include everything that is necessary, but nothing more.**

A function T from the vector space V to the vector space W is a linear transformation if $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(cv_1) = cT(v_1)$ for all $v_1, v_2 \in V$ and $c \in \mathbb{R}$.

5. **Define "diagonalizable". Use complete sentences. Include everything that is necessary, but nothing more.**

The square matrix A is diagonalizable if there exist a diagonal matrix D and an invertible matrix S with $A = SDS^{-1}$.

6. **Define "nonsingular". Use complete sentences. Include everything that is necessary, but nothing more.**

The $n \times n$ matrix A is non-singular if the only vector x in $\mathbb{R}^n$ with $Ax = 0$ is $x = 0$.

7. **Let A be an $n \times n$ matrix. Record eight statements that are equivalent to "the matrix A is invertible".**

1. There is a matrix B with AB equal to the identity matrix and BA equal to the identity matrix.
2. There is a matrix B with AB equal to the identity matrix.
3. There is a matrix B with BA equal to the identity matrix.
4. The null space of A is $\{0\}$.
5. The columns of A are linearly independent.
6. The only solution to $Ax = 0$ is $x = 0$.
7. The columns of A span $\mathbb{R}^n$.
8. The system of equations $Ax = b$ has a solution for all $b \in \mathbb{R}^n$.
9. The columns of A are a basis for $\mathbb{R}^n$.
10. The dimension of the null space of A is zero.
11. The dimension of the column space of A is n.
12. The rank of A is n.
13. The rows of A are linearly independent.
14. The rows of A span the vector space of all row vectors with n entries.
15. The dimension of the row space of A is n.
16. Zero is not an eigenvalue of A.

8. **Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree less than or equal to three. Let $T\colon \mathcal{P}_3 \to \mathbb{R}$ be the linear transformation which is given by $T(p(x)) = \int_{-1}^{1} p(x)dx$. Find a basis for the null space of T.**

The domain of T has dimension 4, the image of T has dimension 1, so the rank-nullity theorem tells us that the null space of T has dimension 3. We complete the problem by exhibiting 3 linearly independent elements of $\mathcal{P}_3$ which are in the null space of T: $\boxed{x, x^3, x^2 - \frac{1}{3}}$.

9. **Let A be a square matrix, v_1 and v_2 be non-zero vectors with $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, where λ_1 and λ_2 are real numbers with $\lambda_1 \neq \lambda_2$. Prove that $\{v_1, v_2\}$ is a linearly independent set of vectors.**

Suppose

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiply both sides of (1) by A to get

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0. \tag{2}$$

Multiply both sides of equation (1) by λ_2 to get

$$c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0. \tag{3}$$

Subtract (2) minus (3) to get

$$c_1(\lambda_1 - \lambda_2)v_1 = 0.$$

The vector v_1 is not zero. If a scalar times v_1 is zero, then the scalar must be zero. Thus, the scalar $c_1(\lambda_1 - \lambda_2) = 0$. But, $(\lambda_1 - \lambda_2)$ is not zero; so, c_1 must be zero. Equation (1) now says that $c_2 v_2 = 0$. The vector v_2 is not zero; so, the scalar c_2 must be zero.

10. **Find an orthogonal basis for the null space of $A = [1 \quad 2 \quad 3 \quad 5]$.**

One basis for the null space of A is

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We apply the Gram-Schmidt orthogonalization process to this basis. Let $u_1 = v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Let

$$u_2' = v_2 - \frac{u_1^{\mathrm{T}} v_2}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.$$

(Notice that $Au_2 = 0$ and $u_1^{\mathrm{T}} u_2 = 0$.) Let

$$u_3' = v_3 - \frac{u_1^{\mathrm{T}} v_3}{u_1^{\mathrm{T}} u_1} u_1 - \frac{u_2^{\mathrm{T}} v_3}{u_2^{\mathrm{T}} u_2} u_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underbrace{\frac{15}{70}}_{\frac{3}{14}} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{14} \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}$$

Let

$$u_3 = \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}.$$

It is easy to check that

$$\boxed{u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -5 \\ -10 \\ -15 \\ 14 \end{bmatrix}}$$

is an orthogonal basis for the null space of A.

11. **Let** $A = \begin{bmatrix} 5 & -2 \\ \frac{28}{3} & -\frac{11}{3} \end{bmatrix}$ **. Find** $\lim_{n\to\infty} A^n$ **.**

We see that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. It follows that $AS = SD$ for $S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. Thus,

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} (SDS^{-1})^n = S \left(\lim_{n\to\infty} D^n \right) S^{-1} = S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 7 & -3 \\ 14 & -6 \end{bmatrix}}.$$

12. **Consider the function** $T\colon \operatorname{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ **which sends a** 2×2 **matrix** A **to the real number** $\det(A)$**. Is** T **a linear transfromation? Explain.**

$\boxed{\text{NO!}}$. Observe that

$$T\left(2\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) = T\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right) = 4 \quad \text{and} \quad 2T\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) = 2;$$

thus, $T(cA)$ is not always equal to $cT(A)$.

13. **Express** $v = \begin{bmatrix}1\\1\\0\end{bmatrix}$ **as a linear combination of** $u_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$**,** $u_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$**,** $u_3 = \begin{bmatrix}-1\\2\\-1\end{bmatrix}$**. (You are welcome to notice that** u_1, u_2, u_3 **form an orthogonal set of vectors.) Check your answer.**

Suppose $v = c_1u_1 + c_2u_2 + c_3u_3$. Multiply both sides by u_1^{T} to see that $2 = 3c_1$; hence, $c_1 = \frac{2}{3}$, Multiply by u_2^{T} to see that $-1 = 2c_2$; hence $c_2 = \frac{-1}{2}$. Multiply by u_3^{T} to see that $1 = 6c_3$; hence $c_3 = \frac{1}{6}$. We check that

$$\frac{2}{3}u_1 - \frac{1}{2}u_2 + \frac{1}{6}u_3 = \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} + \frac{1}{6}\begin{bmatrix}-1\\2\\-1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}4+3-1\\4+0+2\\4-3-1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} = v. \checkmark$$

14. **Let**

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad v_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Let V **be a subspace of** $\mathbb{R}^4$**. Suppose that** $v_1 \in V$**,** $v_2 \in V$**,** $v_3 \notin V$**, and** $v_4 \notin V$**. Do you have enough information to determine the dimension of** V**? Explain very thoroughly.**

$\boxed{\text{NO}}$. The vector space V could have dimension 2. (In this case v_1 and v_2 are a basis for V.) On the other hand, the vector space V could have dimension 3. For example, the vector space V spanned by v_1, v_2, and

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

has dimension 3 and does not contain v_3 or v_4.

15. **Let**

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let V be a subspace of $\mathbb{R}^4$. Suppose that $v_1 \in V$, $v_2 \in V$, $v_3 \in V$, and $v_4 \notin V$. Do you have enough information to determine the dimension of V? Explain very thoroughly.

The vector space V has dimension 3. We have exhibited 3 linearly independent vectors v_1, v_2 and v_3 in V. So $\dim V \geq 3$. On the other hand, V is a subspace of the 4 dimensional vector space $\mathbb{R}^4$; so $\dim V \leq 4$. Finally, if $\dim V$ were equal to 4; then V would have to equal $\mathbb{R}^4$. However, V does not equal $\mathbb{R}^4$ because v_4 is not in V.

16. **Let v_1, v_2, and v_3 be non-zero vectors in $\mathbb{R}^4$. Suppose that $v_i^{\mathrm{T}} v_j = 0$ for all subscripts i and j with $i \neq j$. Prove very thoroughly that v_1, v_2, and v_3 are linearly independent.**

Suppose c_1, c_2, and c_3 are numbers with

$$(*) \qquad c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$

Multiply by v_1^{T} to get

$$c_1 \cdot v_1^{\mathrm{T}} v_1 + c_2 \cdot v_1^{\mathrm{T}} v_2 + c_3 \cdot v_1^{\mathrm{T}} v_3 = 0.$$

The hypothesis tells us that $v_1^{\mathrm{T}} v_2 = 0$ and $v_1^{\mathrm{T}} v_3 = 0$. So, $c_1 \cdot v_1^{\mathrm{T}} v_1 = 0$. The hypothesis also tells us that v_1 is not zero; from which it follows that $v_1^{\mathrm{T}} v_1 \neq 0$. We conclude that $c_1 = 0$. Multiply (*) by v_2^{T} to see that $c_2 \cdot v_2^{\mathrm{T}} v_2 = 0$; hence, $c_2 = 0$, since the number $v_2^{\mathrm{T}} v_2 \neq 0$. Multiply (*) by v_3^{T} to conclude that $c_3 = 0$. We have shown that each c_i MUST be zero. We conclude that v_1, v_2, and v_3 are linearly independent.