Math 544, Exam 3, Spring 2011

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **8** problems on **TWO SIDES**. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. **Write in complete sentences.** I will post the solutions on my website.

Please Note: The next quiz will be **Thursday**.

Please Note: In this exam, if V is a subset of $\mathbb{R}^n$ for some n, then the phrases: "V is a subspace of $\mathbb{R}^n$" and "V is a vector space" have exactly the same meaning.

1. (7 points) **Define "dimension". Use complete sentences. Include everything that is necessary, but nothing more.**

The *dimension* of the vector space V is the number of vectors in a basis for V.

2. (7 points) **Define "basis". Use complete sentences. Include everything that is necessary, but nothing more.**

A *basis* for the vector space V is a linearly independent subset of V which also spans V.

3. (6 points) **Define "subspace of $\mathbb{R}^n$". Use complete sentences. Include everything that is necessary, but nothing more.**

The subset V of $\mathbb{R}^n$ is a *subspace of* $\mathbb{R}^n$ if

(1) the zero vector of $\mathbb{R}^n$ is an element of V,
(2) V is closed under addition (that is, if v_1 and v_2 are in V, then $v_1 + v_2$ is in V), and
(3) V is closed under scalar multiplication (that is, if v is an elemnent of V and c is a constant, then cv is an element of V).

4. (6 points) **State the Four Theorems about Dimension. Use complete sentences. Include everything that is necessary, but nothing more.**

Theorem 1. If V is a subsapce of R^n , then every basis for V has the same number of vectors.

Theorem 2. If V is a subsapce of R^n , then every linearly independent subset in V is part of a basis for V .

Theorem 3. If V is a subsapce of R^n , then every finite spanning set for V contains a basis for V .

Theorem 4. If A is a matrix, then the dimension of the column space of A plus the dimension of the null space of A is equal to the number of columns of A .

5. (6 points) **Let**

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \middle| ab = 0 \right\}.$$

Is V a vector space? If yes, prove your answer. If no, give an example which shows why V is not a vector space. Record a thorough answer. Use complete sentences.

NO!. The set V is not closed under addition. Indeed, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are in V , but $v_1 + v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is not in V .

6. (6 points) **Let W be a subspace of $\mathbb{R}^n$ and A be an $m \times n$ matrix. Let**

$$V = \{Aw \mid w \in W\}.$$

Is V a vector space? If yes, prove your answer. If no, give an example which shows why V is not a vector space. Record a thorough answer. Use complete sentences.

YES!.

"The cute proof": Pick a basis $w_1, \dots, w_r$ for W . Let B be the $n \times r$ matrix whose columns are $w_1, \dots, w_r$. Then V is the column space of AB . We proved that the column space of every matrix is a vector space.

"The straightforward proof": We show that V satisfies the axioms of problem 3.

0 is in V. The set W is a vector space, so 0 is in W and $A0$, which is equal to 0, is in V.

V is closed under addition. Take v_1 and v_2 from V. The definition of V says that there exist w_1 and w_2 in W with $v_i = Aw_i$, for $1 \le i \le 2$. The set W is a vector space; so $w_1 + w_2$ is in W. It follows that $A(w_1 + w_2)$ is in V. On the other hand, $A(w_1 + w_2) = Aw_1 + Aw_2 = v_1 + v_2$ because matrix multiplication distributes over addition. Thus, $v_1 + v_2$ is in V.

V is closed under scalar multiplication. Take v_1 from V and a constant c. The definition of V says that there exist w_1 in W with $v_1 = Aw_1$. The set W is a vector space; so cw_1 is in W. It follows that $A(cw_1)$ is in V. On the other hand, $A(cw_1) = cA(w_1) = cv_1$ by the properties of constants, Thus cv_1 is in V.

7. (6 points) **Let A and B be $n \times n$ matrices with B non-singular. Does the column space of BA have to equal the column space of A? Prove your answer very thoroughly. Use complete sentences.**

NO!. Take $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We see that the column space of A is $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$. On the other hand, $BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, which has column space $\left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} \middle| b \in \mathbb{R} \right\}$. In this example, the column space of A is different than the column space of BA.

8. (6 points) **Let A and B be $n \times n$ matrices with B non-singular. Does the dimension of the column space of of BA have to equal the dimension of the column space of A? Prove your answer very thoroughly. Use complete sentences.**

YES!.

"The proof we gave as answer to a Homework problem". We first show that BA and A have the same null space.

We prove that the nullspace of A is contained in the null space of BA. Take v in the null space of A. So, $Av = 0$. It follows that $BAv = B0 = 0$ and v is in the null space of BA.

Now, we prove that the nullspace of BA is contained in the null space of A. Take v in the null space of BA. So, $BAv = 0$. The matrix B is nonsingular. If

$Bw = 0$ for some vector w, then w must be 0. We have $B(Av) = 0$. It follows that $Av = 0$ and v is in the null space of A.

We conclude this proof. The matrices A and BA have the same null space; so they have the same nullity. They also have the same number of columns. The rank-nullity Theorem guarantees that they have the same rank.

"A different proof". Let $w_1 \dots, w_r$ be a basis for the column space of A. So there exist vecotrs $v_1, \dots, v_r$ in $\mathbb{R}^n$, with $Av_i = w_i$, for $1 \le i \le r$. We will show that $BAv_1, \dots, BAv_r$ is a basis for the column space of BA. (This will show that r, which is the dimension of the column space of A, is also equal to the dimension of the column space of BA.) We first see that each of the vectors $BAv_1, \dots, BAv_r$ is in the column space of BA.

We now show that $BAv_1, \dots, BAv_r$ span the column space of BA. Take u in the column space of BA. It follows that $u = BAz$ for some vector $z \in \mathbb{R}^n$. Observe that Az is in the column space of A and $w_1, \dots, w_r$ is a basis for the column space of A. It follows that there are constants α_i with $Az = \sum_{i=1}^{r} \alpha_i w_i$. Multiply B to see that

$$u = BAz = \sum_{i=1}^{r} \alpha_i B w_i = \sum_{i=1}^{r} \alpha_i BAv_i.$$

Finally, we show that $BAv_1, \dots, BAv_r$ are linearly independent. Suppose $\alpha_1, \dots, \alpha_r$ are constants with $\sum_{i=1}^{r} \alpha_i BAv_i$. It follows that $B(\sum_{i=1}^{r} \alpha_i Av_i) = 0$. The matrix B is non-singular, so $\sum_{i=1}^{r} \alpha_i Av_i = 0$. The most recent equation is the same as $\sum_{i=1}^{r} \alpha_i w_i = 0$. The vectors $w_1, \dots, w_r$ are linearly independent so $\alpha_1 = \dots = \alpha_r = 0$.