Math 544, Exam 2, Fall 2009

Write your answers as legibly as you can.

There are 7 problems. Problem 7 is worth 22 points. Each of the other problems is worth 13 points. The exam is worth a total of 100 points. SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible. **No Calculators.**

I will post the solutions on my website shortly after the class is finished.

1. **Let $U \subseteq V$ be subspaces of $\mathbb{R}^n$ with $\dim U = \dim V$. Do U and V HAVE to be equal? If yes, prove your answer. If no, give an example.**

YES. Let $u_1, \dots, u_r$ be a basis for U. Thus $u_1, \dots, u_r$ is a linearly independent set in the vector space V. One of the dimension theorems tells us that $u_1, \dots, u_r$ is the begining of a basis for V; that is, we may adjoin more vectors to this list, if necessary, to get a basis for V. However every basis for V has $\dim V$ vectors and $\dim V = \dim U = r$. Thus, $u_1, \dots, u_r$ is already a basis for V. Thus U and V are both spanned by $u_1, \dots, u_r$ and $U = V$.

2. **Let A and B be $n \times n$ matrices. Does the null space of AB HAVE to be a subset of the null space of A? If yes, prove your answer. If no, give an example.**

NO. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

So,

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We see that $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the null space of AB, because $ABv = 0$; but v is not in the null space of A, because $Av = v \neq 0$.

3. **Define "null space". Use complete sentences. Include everything that is necessary, but nothing more.**

The null space of the matrix A is the set of all column vectors x with $Ax = 0$.

4. **Define "dimension". Use complete sentences. Include everything that is necessary, but nothing more.**

The dimension of the vector space V is the number of vectors in a basis for V.

5. **Let**

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, 2x_1 + 3x_3 - 4x_3 = 5 \right\}.$$

Is V a vector space? Explain thoroughly.

NO. The vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in V.

6. **Let** $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ **and** $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ **be fixed elements of** $\mathbb{R}^3$**, and let**

$$V = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, a^{\mathrm{T}} x = 0 \textbf{ and } b^{\mathrm{T}} x = 0 \right\}.$$

Is V a vector space? Explain thoroughly.

YES. The set V is the null space of

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

The null space of any matrix is a vector space.

7. **Let A be the matrix**

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 & 4 \\ 1 & 3 & 4 & 3 & 6 \\ 2 & 6 & 8 & 5 & 10 \end{bmatrix}$$

(a) Find a basis for the null space of A.
(b) Find a basis for the column space of A.
(c) Find a basis for the row space of A.
(d) Write each column of A as a linear combination of your answer to (b).
(e) Write each row of A as a linear combination of your answer to (c).

Apply elementary row operations $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - 2R_1$ to get

$$\begin{bmatrix} 1 & 3 & 4 & 2 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Apply elementary row operations $R_1 \mapsto R_1 - 2R_2$ and $R_3 \mapsto R_3 - R_2$ to get

$$\begin{bmatrix} 1 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nullspace of A is the set of all vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

where x_2, x_3, and x_5 are free to take any value.

(a) It follows that the vectors

$$\boxed{v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$

form a basis for the null space of A.

(b) The vectors

$$\boxed{A_{*,1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A_{*,4} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}}$$

form a basis for the column space of A.

(c) The vectors

$$\boxed{w_1 = \begin{bmatrix} 1 & 3 & 4 & 0 & 0 \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \end{bmatrix}}$$

form a basis for the row space of A.

(d) We see that

$$\boxed{\begin{array}{c} A_{*,1} = A_{*,1} \\ A_{*,2} = 3A_{*,1} \\ A_{*,3} = 4A_{*,1} \\ A_{*,4} = A_{*,4} \\ A_{*,5} = 2A_{*,4} \end{array}}.$$

(e) We see that

$$\boxed{\begin{array}{c} A_{1,*} = 1w_1 + 2w_2 \\ A_{2,*} = w_1 + 3w_2 \\ A_{3,*} = 2w_1 + 5w_2 \end{array}}.$$