Math 544, Exam 1, Spring, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Solve the system of equations $Ax = b$, where**

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix}.$$

If $Ax = b$ has more than one solution, then give the general solution and four particular solutions.

We consider the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 9 & 13 \\ 1 & 2 & 3 & 2 & 13 & 20 \\ 2 & 4 & 6 & 3 & 22 & 33 \end{array}\right]$$

Replace row 2 with row 2 minus row 1.
Replace row 3 with row 3 minus 2 times row 1.
Obtain

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 9 & 13 \\ 0 & 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & 4 & 7 \end{array}\right]$$

Replace row 1 with row 1 minus row 2, and
replace row 3 with row 3 minus row 2 to obtain

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The most recent augmented matrix corresponds to the equations:

$$\begin{array}{rrl} x_1 + 2x_2 + 3x_3 & & +5x_5 = 6 \\ & x_4 & +4x_5 = 7 \end{array}$$

These equations tell us that x_1 and x_4 are dependent variables and x_2, x_3, and x_5 are free to take any value. The general solution of $Ax = b$ is

$$\begin{cases} x_1 = & 6 & -2x_2 & -3x_3 & -5x_5 \\ x_2 = & & x_2 & & \\ x_3 = & & & x_3 & \\ x_4 = & 7 & & & -4x_5 \\ x_5 = & & & & x_5, \\ \multicolumn{5}{l}{\text{where } x_2, x_3, \text{ and } x_5, \text{ are free to take any value.}} \end{cases}$$

One could also write that the general solution of $Ax = b$ is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \;\middle|\; \text{where } x_2, x_3, \text{ and } x_5, \text{ are free to take any value.} \right\}$$

When $x_2 = x_3 = x_5 = 0$, then the solution is

$$v_1 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}.$$

When $x_2 = 1$ and $x_3 = x_5 = 0$, then the solution is

$$v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 7 \\ 0 \end{bmatrix}.$$

When $x_3 = 1$ and $x_2 = x_5 = 0$, then the solution is

$$v_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 7 \\ 0 \end{bmatrix}.$$

When $x_5 = 1$ and $x_2 = x_3 = 0$, then the solution is

$$v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

We check that

$$Av_1 = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix} . \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix} . \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix} . \checkmark$$

$$Av_4 = \begin{bmatrix} 1 & 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 2 & 13 \\ 2 & 4 & 6 & 3 & 22 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 33 \end{bmatrix} . \checkmark$$

We have verified that

$$\boxed{v_1 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 7 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}}$$

are particular solutions of $Ax = b$.

(2) **Consider the system of equations $Ax = b$, where**

$$A = \begin{bmatrix} 1 & a^2 - 1 \\ 2 & a^2 - 1 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} 3 - a \\ 5 - a \end{bmatrix}.$$

(a) **Find all values of a for which the system of equations has no solution.**

(b) **Find all values of a for which the system of equations has exactly one solution.**

(c) **Find all values of a for which the system of equations has an infinite number of solutions.**

We consider the augmented matrix

$$\left[\begin{array}{cc|c} 1 & a^2-1 & 3-a \\ 2 & a^2-1 & 5-a \end{array}\right]$$

Replace row 2 with row 2 -2 row 1 to obtain:

$$\left[\begin{array}{cc|c} 1 & a^2-1 & 3-a \\ 0 & -a^2+1 & -1+a \end{array}\right] \tag{1}$$

If $a^2 - 1$ is not zero, then the system of equations $Ax = b$ has a unique solution.

In other words,

$$\boxed{\text{if } a \text{ is not equal to } 1 \text{ or } -1, \text{ then } Ax = b \text{ a unique solution.}}$$

If $a = 1$, then (1) is

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The solution set is $x_1 = 0$ and x_2 is free to take any value. Thus,

$$\boxed{\text{if } a = 1, \text{ then } Ax = b \text{ has an infinite number of solutions.}}$$

If $a = -1$, then (1) is

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 0 & -2 \end{array}\right]$$

The corresponding system of equations has no solution. Thus,

$$\boxed{\text{if } a = -1, \text{ then } Ax = b \text{ has no solution.}}$$

(3) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \ldots, v_p$ in $\mathbb{R}^m$ are <u>linearly independent</u> if the only numbers $c_1, \ldots, c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

(4) **Suppose v_1, v_2, and v_3 are three vectors in $\mathbb{R}^m$, for some m, with v_1, v_2 linearly independent, v_1, v_3 linearly independent, and v_2, v_3 linearly independent. Do the vectors v_1, v_2, v_3 have to be linearly independent? If the answer is yes, prove it. If the answer is no, give a counterexample.**

Of course not. Take

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{in } \mathbb{R}^2.$$

It is easy to see that the vectors v_1, v_2 are linearly independent; the vectors v_1, v_3 are linearly independent; and the vectors v_2, v_3 are linearly independent. However, the vectors v_1, v_2, v_3 are linearly dependent by the short-fat theorem which states that if $m < p$, and $v_1, \ldots, v_p$ are vectors in $\mathbb{R}^m$, then $v_1, \ldots, v_p$ are linearly dependent.

(5) **Let v_1, v_2, and v_3 be non-zero vectors in $\mathbb{R}^m$, for some m. Suppose that $v_i^{\mathrm{T}} v_j = 0$ for all subscripts i and j with $i \neq j$. Do the vectors v_1, v_2, v_3 have to be linearly independent? If the answer is yes, prove it. If the answer is no, give a counterexample.**

Suppose c_1, c_2, and c_3 are numbers with

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0. \tag{2}$$

Multiply by v_1^{T} to get

$$c_1 \cdot v_1^{\mathrm{T}} v_1 + c_2 \cdot v_1^{\mathrm{T}} v_2 + c_3 \cdot v_1^{\mathrm{T}} v_3 = 0.$$

The hypothesis tells us that $v_1^{\mathrm{T}} v_2 = 0$ and $v_1^{\mathrm{T}} v_3 = 0$. So, $c_1 \cdot v_1^{\mathrm{T}} v_1 = 0$. The hypothesis also tells us that v_1 is not zero; from which it follows that $v_1^{\mathrm{T}} v_1 \neq 0$. We conclude that $c_1 = 0$. Multiply (2) by v_2^{T} to see that $c_2 \cdot v_2^{\mathrm{T}} v_2 = 0$; hence, $c_2 = 0$, since the number $v_2^{\mathrm{T}} v_2 \neq 0$. Multiply (2) by v_3^{T} to conclude that $c_3 = 0$. We have shown that each c_i MUST be zero. We conclude that v_1, v_2, and v_3 are linearly independent.