1

$$\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 1 & 2 & 3 & 9 \\ 1 & 2 & 3 & 2 & 4 & 6 & 13 \\ 1 & 2 & 3 & 3 & 6 & 9 & 17 \end{array}\right] \quad \left[\begin{array}{cccccc|c} 1 & 2 & 3 & 1 & 2 & 3 & 9 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 & 4 & 6 & 8 \end{array}\right]$$

$R2 \mapsto R2 - R1$
$R3 \mapsto R3 - R1$

$R1 \mapsto R1 - R2$
$R3 \mapsto R3 - R2$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 &= 5 - 2x_2 - 3x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= 4 - 2x_5 - 3x_6 \\ x_5 &= x_5 \\ x_6 &= x_6 \end{aligned}$$

The General Solution

<u>specific solution 1</u> Take $x_2 = x_3 = x_5 = x_6 = 0$

$$v_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} \quad \text{we check } Av_1 = \begin{bmatrix} 5+4 \\ 5+8 \\ 5+12 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 17 \end{bmatrix} = b \checkmark$$

<u>specific sol 2</u> Take $x_2 = 1$ $x_3 = x_5 = x_6 = 0$

$$v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} \quad \text{we check } Av_2 = \begin{bmatrix} 3+2+4 \\ 3+2+8 \\ 3+2+12 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 17 \end{bmatrix} = b \checkmark$$

<u>specific sol 3</u> Take $x_3 = 1$ $x_2 = x_5 = x_6 = 0$

$$v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} \quad \text{we check } Av_3 = \begin{bmatrix} 2+3+4 \\ 2+3+8 \\ 2+3+12 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 17 \end{bmatrix} = b \checkmark$$

Specific Sol 4   Take $X_5=1$ $X_2=X_3=X_6=0$

$$V_4=\begin{bmatrix}5\\0\\0\\2\\1\\0\end{bmatrix} \quad \text{Check } AV_4=\begin{bmatrix}5+2+2\\5+4+4\\5+6+6\end{bmatrix}=\begin{bmatrix}9\\13\\17\end{bmatrix} \checkmark$$

Specific Sol 5   Take $X_6=1$ $X_2=X_3=X_5=0$

$$V_5=\begin{bmatrix}5\\0\\0\\1\\0\\1\end{bmatrix} \quad \text{Check } AV_5=\begin{bmatrix}5+1+3\\5+2+6\\5+3+9\end{bmatrix}=\begin{bmatrix}9\\13\\17\end{bmatrix} \checkmark$$

2

$$\left[\begin{array}{cc|c}1 & a & 1\\ a-1 & 6 & 2\end{array}\right] \qquad \left[\begin{array}{cc|c}1 & a & 1\\ 0 & 6-(a-1)a & 2-(a-1)\end{array}\right] \quad (*)$$

$R2 \mapsto R2+(a-1)R1$

If $6-(a-1)a$ is not zero, then the system of equations has a unique solution.

Well $6-(a-1)a=0$ when $6-a^2+a=0$

so $a^2-a-6=0$

so $(a-3)(a+2)=0$

so $a=3$ or $a=-2$

$\boxed{\text{(b) If } a\neq 3 \text{ or } -2\text{, then the system of equations has a unique solution.}}$

If $a=3$, then $(*)$ is

$$\left[\begin{array}{cc|c}1 & 3 & 1\\ 0 & 0 & 0\end{array}\right]$$

In this case the system of equations has an ~~unique solution~~ infinite number of solutions.

$\boxed{\text{(c) If } a=3\text{, then the system has an infinite number of solutions.}}$

If $a=-2$, then $\#$ is

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 5 \end{array}\right]$$

In this case the system has no solutions.

(a) If $a=-2$, then the system has no solution

(3) We find all c_1, c_2, c_3 with

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

That is we solve

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \#\#$$

$R2 \mapsto R2 - R1$
$R3 \mapsto R3 - R1$
$R4 \mapsto R4 - R1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

$R1 \mapsto R1 + \frac{1}{2} R2$
$R4 \mapsto R4 - R2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$R1 \mapsto R1 + \frac{1}{2} R3$
$R4 \mapsto R4 - R3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R2 \mapsto -\frac{1}{2} R2$
$R3 \mapsto \frac{1}{2} R3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that $c_1 = c_2 = c_3 = 0$ is the <u>only</u> solution of $\#\#$. We conclude v_1, v_2, v_3 are linearly independent vectors

(4) $\boxed{\text{No}}$ Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

We see that A and B both are symmetric matrices

but $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not a symmetric matrix.

(5) To decide if w_1, w_2, w_3 are linearly independent

We find all numbers c_1, c_2, c_3 with

$c_1 w_1 + c_2 w_2 + c_3 w_3 = 0 \longleftarrow (**)$

I.e., $c_1 (v_1 + v_2 + v_3) + c_2 (v_1 + v_3) + c_3 (v_2 + v_3) = 0$

So $(c_1 + c_2) v_1 + (c_1 + c_3) v_2 + (c_1 + c_2 + c_3) v_3 = 0$

The vectors v_1, v_2, v_3 are linearly independent by hypothesis. Thus,

$c_1 + c_2 = 0$
$c_1 + c_3 = 0$
$c_1 + c_2 + c_3 = 0$

Now we have some equations. We can solve these equations

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (*)$$

$R2 \mapsto R2 - R1$
$R3 \mapsto R3 - R1$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$R1 \mapsto R1 + R2$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$R1 \mapsto R1 - R3$
$R2 \mapsto R2 - R3$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The only solution of $(*)$ is $c_1 = c_2 = c_3 = 0$

So the <u>only</u> numbers c_1, c_2, c_3 with $(**)$ are $c_1 = c_2 = c_3 = 0$

$\therefore$ w_1, w_2, w_3 <u>Are</u> linearly independent so Yes

6 No. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

We see $(A-B)(A+B) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

We also see $A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Thus $(A-B)(A+B) \neq A^2 - B^2$