Math 544, Exam 1, Summer 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **7** problems on **TWO SIDES**. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences. I will post the solutions on my website shortly after the exam is finished.

1. (8 points) **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.** $\boxed{CIRCLE}$ **your answer.**

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ 1 & 2 & 2 & 10 & 1 \\ 1 & 2 & 1 & 7 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 15 \\ 20 \\ 21 \end{bmatrix}.$$

We apply row operations to

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 7 & 1 & 15 \\ 1 & 2 & 2 & 10 & 1 & 20 \\ 1 & 2 & 1 & 7 & 2 & 21 \end{array}\right]$$

Replace row 2 by row 2 minus row 1. Replace row 3 by row 3 minus row 1.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 7 & 1 & 15 \\ 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Replace row 1 by row 1 minus row 2.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 1 & 10 \\ 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Replace row 1 by row 1 minus row 3.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 4 \\ 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

The General solution of $Ax = b$ is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix},}$$

where x_2 and x_4 are arbitrary.

One particular solution occurs $x_2 = x_4 = 0$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \\ 6 \end{bmatrix}.$$

We verify that

$$\begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ 1 & 2 & 2 & 10 & 1 \\ 1 & 2 & 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \\ 21 \end{bmatrix}.$$

A second solution occurs when $x_2 = 1$ and $x_4 = 0$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 6 \end{bmatrix}.$$

We verify that

$$\begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ 1 & 2 & 2 & 10 & 1 \\ 1 & 2 & 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \\ 21 \end{bmatrix}.$$

A third solution occurs when $x_2 = 0$ and $x_4 = 1$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}.$$

We verify that

$$\begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ 1 & 2 & 2 & 10 & 1 \\ 1 & 2 & 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \\ 21 \end{bmatrix}.$$

2. (7 points) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

3. (7 points) **Let A and B be 2×2 matrices with A not equal to the zero matrix and $BA = A^2$. Does B have to equal A? If yes, prove your answer. If no, give a counterexample.**

$\boxed{\text{NO}}$. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$. We see that A is not the zero matrix, A is not equal to B, but A^2 and BA both are equal to $\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$.

4. (7 points) **Let A and B be 2×2 symmetric matrices. Does the product AB have to be a symmetric matrix? If yes, prove your answer. If no, give a counterexample.**

$\boxed{\text{NO}}$. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$. We see that A and B are symmetric; but $AB = \begin{bmatrix} 7 & 3 \\ 11 & 6 \end{bmatrix}$ is not symmetric.

5. (7 points) **Let** v_1, v_2, v_3 **be linearly independent vectors in** $\mathbb{R}^3$**. Define** v_1^*, v_2^*, v_3^* **to be the vectors**

$$v_1^* = \begin{bmatrix} v_1 \\ 1 \end{bmatrix}, \quad v_2^* = \begin{bmatrix} v_2 \\ 1 \end{bmatrix}, \quad v_3^* = \begin{bmatrix} v_3 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^4$**. Do the vectors** v_1^*, v_2^*, v_3^* **have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

$\boxed{\text{The vectors } v_1^*, v_2^*, v_3^* \text{ are linearly independent.}}$ If c_1, c_2, and c_3 are numbers with $c_1v_1^* + c_2v_2^* + c_3v_3^* = 0$, then in particular, by looking only at the top three rows, we see that $c_1v_1 + c_2v_2 + c_3v_3 = 0$. However the vectors v_1, v_2, v_3 are linearly independent; so, c_1, c_2, and c_3 are required to be zero.

6. (7 points) **Consider the system of linear equations.**

$$\begin{aligned} x_1 + (a-1)x_2 &= 4 \\ ax_1 + \quad 6x_2 &= 12. \end{aligned}$$

(a) **Which values for** a **cause the system to have no solution?**
(b) **Which values for** a **cause the system to have exactly one solution?**
(c) **Which values for** a **cause the system to have an infinite number of solutions?**

Explain thoroughly.

Consider the augmented matrix

$$\left[\begin{array}{cc|c} 1 & a-1 & 4 \\ a & 6 & 12 \end{array}\right].$$

Replace row 2 by row 2 minus a times row 1 to get:

$$\left[\begin{array}{cc|c} 1 & a-1 & 4 \\ 0 & 6-a(a-1) & 12-4a \end{array}\right].$$

Multiply row 2 by -1 to get

$$\left[\begin{array}{cc|c} 1 & a-1 & 4 \\ 0 & a^2-a-6 & 4a-12 \end{array}\right],$$

which is the same as

$$\left[\begin{array}{cc|c} 1 & a-1 & 4 \\ 0 & (a-3)(a+2) & 4(a-3) \end{array}\right].$$

We see that if a is any number other than 3 or -2, then the system of equations has a unique solution. (In this case the bottom row tells the value of x_2 and the top row tells the value of x_1.) If $a = 3$, then the bottom row is completely zero and the solution set is the line described by the top equation. If $a = -2$, then the system of equations has no solution because 0 is never equal to -20. To summarize:

(a) If $a = -2$, then the system of equations has no solution.
(b) If $a \neq -2$ and $a \neq 3$, then the system of equations has exactly one solution.
(b) If $a = 3$, then the system of equations has an infinite number of solutions.

7. (7 points) **Are the vectors**

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

linearly independent? Explain thoroughly.

We find all c_1, c_2, c_3 with

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

by applying Elementary Row Operations to

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Replace row 2 with row 2 minus 2 times row 1. Replace row 3 with row 3 minus 3 times row 1.

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

Replace row 2 with minus one third of row 2:

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix}$$

Replace row 1 with row 1 minus 4 row 2. Replace row 3 with row 3 plus 6 row 2:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, c_3 is free to take any value and $c_1 = c_3$ and $c_2 = -2c_3$. In particular, take $c_3 = 1$. It is indeed true that $v_1 - 2v_2 + v_3 = 0$. Thus,

$$\boxed{v_1,\ v_2,\ v_3 \text{ is a linearly dependent collection of vectors.}}$$