Math 544, Exam 1, Spring 2011 Solutions

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **9** problems on **TWO SIDES**. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences. I will post the solutions on my website shortly after the exam is finished.

1. (10 points) **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.** $\boxed{CIRCLE}$ **your answer.**

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix}.$$

We apply row operations to

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 1 & 8 & 3 \\ 1 & 4 & 5 & 2 & 10 & 5 \\ 3 & 12 & 15 & 4 & 26 & 11 \end{array}\right].$$

Replace $R2 \mapsto R2 - R1$ and $R3 \mapsto R3 - 3R1$ to get

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 1 & 8 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right].$$

Replace $R1 \mapsto R1 - R2$ and $R3 \mapsto R3 - R2$ to get

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution of the system of equations is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where x_2, x_3, and x_5 are free to take any value.

To obtain particular solutions of the system of equations, take $x_2 = x_3 = x_5 = 1$ to obtain v_1 ; $x_2 = 1$, $x_3 = x_5 = 0$ to obtain v_2 ; $x_2 = x_5 = 0$, $x_3 = 1$ to obtain v_3 and $x_2 = x_3 = 0$, $x_5 = 1$ to obtain v_4 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We check

$$Av_1 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \checkmark$$

and

$$Av_4 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b. \checkmark$$

2. (5 points) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

3. (5 points) **Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.**

The square matrix A is <u>non-singular</u> if the **only** column vector x with $Ax = 0$ is $x = 0$.

4. (5 points) **State the result about the linear dependence or linear independence p vectors in $\mathbb{R}^m$. Include everything that is necessary, but nothing more.**

If $p > m$, then every set of p vectors in $\mathbb{R}^m$ is linearly dependent. (We have called this the short-wide theorem.)

5. (5 points) **Let A be a non-singular $n \times n$ matrix and let b be an element of $\mathbb{R}^n$. Prove that $Ax = b$ has at least one solution. (I want a complete proof. The answer "We did this in class" is not acceptable.)**

Let A_i denote the i^{th} column of A. Consider the $n+1$ column vectors $A_1, \dots, A_n, b$ from $\mathbb{R}^n$. The "Short-Wide Theorem" tells us that these $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent. Thus there exists numbers $c_1, \dots, c_{n+1}$, at least one of which is non-zero, such that $c_1 A_1 + \cdots + c_n A_n + c_{n+1} b = 0$. We observe that c_{n+1} is not zero because otherwise $c_1 A_1 + \cdots + c_n A_n = 0$ is a non-trivial linear combination of the columns of A which equals zero. This would violate the hypothesis that A is non-singular. Thus,

$$A \begin{bmatrix} \frac{-c_1}{c_{n+1}} \\ \vdots \\ \frac{-c_n}{c_{n+1}} \end{bmatrix} = b.$$

6. (5 points) **Let A, B, and C be 2×2 matrices with A not equal to the zero matrix and $BA = CA$. Does B have to equal C? If yes, prove your answer. If no, give a counterexample.**

NO. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix}$. We see that $B \neq C$ but BA and CA are both equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

7. (5 points) **Let A and B be 2×2 matrices. Does $(A+B)(A-B)$ have to equal $A^2 - B^2$? If yes, prove your answer. If no, give a counterexample.**

N0. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. We see that $A^2 = B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $A^2 - B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We also see that $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A - B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and

$$(A+B)(A-B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq A^2 - B^2.$$

8. (5 points) **Let A and B be $n \times n$ matrices. How is $(AB)^{\mathrm{T}}$ related to the product of A^{T} and B^{T}? Prove that your answer is correct.**

We prove that $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$. The matrices $(AB)^{\mathrm{T}}$ and $B^{\mathrm{T}} A^{\mathrm{T}}$ both are $n \times n$ matrices. We prove that the corresponding entries are equal. Let $A_{i,j}$ be the entry of A in row i and column j. The entry in row r column c of $(AB)^{\mathrm{T}}$ is

$$[(AB)^{\mathrm{T}}]_{r,c} = (AB)_{c,r} = \sum_{j=1}^{n} A_{c,j} B_{j,r} = \sum_{j=1}^{n} B_{j,r} A_{c,j}$$

$$= \sum_{j=1}^{n} (B^{\mathrm{T}})_{r,j} (A^{\mathrm{T}})_{j,c} = [(B^{\mathrm{T}})(A^{\mathrm{T}})]_{r,c}$$

and this is the entry in row r and column c of $(B^{\mathrm{T}})(A^{\mathrm{T}})$. The first equality is the definition of transpose. The second equality is the definition of matrix product. The symbols $A_{c,j}$ and $B_{j,r}$ represent numbers and numbers commute under multiplication. This explains the third equality. The fourth equlaity is the definition of transpose, again. The fifth equality is the definition of matrix multiplication, again.

9. (5 points) Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^4$ with v_1, v_2, v_3 linearly dependent. Do v_1, v_2, v_3, v_4 have to be linearly dependent? If yes, prove your answer. If no, give a counterexample.

YES. The first sentence guarantees that there are numbers a_1, a_2, a_3, at least one of which is non-zero, with $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$. Thus, we have numbers $a_1, a_2, a_3, 0$, at least one of which is not zero, and $a_1 v_1 + a_2 v_2 + a_3 v_3 + 0 v_4 = 0$. We conclude that the vectors v_1, v_2, v_3, v_4 are linearly dependent.