MATH 544, 1998, FINAL EXAM

PRINT Your Name:________________________________

There are 17 problems on 10 pages. The exam is worth 200 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible. **NO CALCULATORS.**

1. (20 points) Let A be an $n \times n$ matrix. List 8 statements that are equivalent to the statement " A is nonsingular".

2. (10 points) Define "vector space".

3. (10 points) Define "eigenvalue".

4. (10 points) Solve the system of equations which corresponds to the following augmented matrix:
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

5. (10 points) Solve the system of equations which corresponds to the following augmented matrix:
$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right].$$

6. (10 points) Solve the system of equations which corresponds to the following augmented matrix:
$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right].$$

7. (20 points) Let
$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 & 0 & 5 & 0 \\ 1 & 0 & 2 & 3 & 4 & 0 & 11 & 0 \\ 1 & 0 & 2 & 3 & 4 & 0 & 11 & 1 \end{bmatrix}.$$

   (a) Find a basis for the row space of A.
   (b) Find a basis for the column space of A.
   (c) Find a basis for the null space of A.
   (d) What is the dimension of the null space of A?
   (e) What is the dimension of the column space of A?

8. (15 points) Let
$$A = \begin{bmatrix} 19 & 30 \\ -9 & -14 \end{bmatrix}.$$

   Find a matrix B with $B^2 = A$.

9. (10 points) Let W be the subspace of $\mathbb{R}^4$ which is spanned by

$$w_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find an orthogonal set which forms a basis for W.

10. (10 points) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

11. (15 points) True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.
   (a) If A and B are 2×2 matrcies, then the null space of B is contained in the null space of AB.
   (b) If A and B are 2×2 matrcies, then the null space of B is contained in the null space of BA.

12. (10 points) Let A be a matrix. Suppose that v_1 and v_2 are non-zero vectors and λ_1 and λ_2 are numbers with $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$, and $\lambda_1 \neq \lambda_2$. PROVE that v_1 and v_2 are linearly independent.

13. (10 points) Let T be the linear transformation of $\mathbb{R}^2$ which fixes the origin, but rotates the plane in the counter clockwise direction by $\pi/4$ radians. Find the matrix M with $T(v) = Mv$ for all $v \in \mathbb{R}^2$.

14. (10 points) Is

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \middle| \; x_1 \text{ and } x_2 \text{ are real numbers} \right\}$$

a vector space? If so, explain why. If not, give an example to show that one of the rules of vector space fails to hold.

15. (10 points) Is the function $F\colon \mathbb{R}^3 \to \mathbb{R}^2$, which is defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + 3x_2 - 2x_3 \end{bmatrix},$$

a linear transformation? If so, explain why. If not, give an example to show that one of the rules of linear transformation fails to hold.

16. (10 points) Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ with

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\2\\3\end{bmatrix}\quad\text{and}\quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}1\\-2\\3\end{bmatrix}.$$

Find a matrix A with $T(v)=Av$ for all $v\in\mathbb{R}^2$.

17. (10 points) Let W be the subspace of $\mathbb{R}^4$ which is spanned by

$$\begin{bmatrix}1\\3\\-1\\0\end{bmatrix},\ \begin{bmatrix}2\\1\\1\\-2\end{bmatrix},\ \begin{bmatrix}-1\\1\\-2\\1\end{bmatrix},\ \begin{bmatrix}1\\2\\2\\2\end{bmatrix}.$$

Find a basis for W .