PRINT Your Name:____________________________________

Quiz for April 15, 2010

The quiz is worth 5 points. **Remove EVERYTHING from your desk except this quiz and a pen or pencil. SHOW every step**. Express your work in a neat and coherent manner. $\boxed{\text{BOX}}$ your answer.

Use the method of Laplace transforms to find a non-trivial solution of

$$tx'' + (t-2)x' + x = 0, \quad x(0) = 0.$$

You might find of the following formulas to be useful:

$\mathcal{L}(\sin kt) = \frac{k}{s^2+k^2}$
$\mathcal{L}(\cos kt) = \frac{s}{s^2+k^2}$
If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$
$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$.

ANSWER: Let $\mathcal{L}(x) = X$. It follows that

$$\begin{aligned}
&\mathcal{L}(x') = s\mathcal{L}(x) - x(0) = sX \\
&\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s^2X - x'(0) \\
&\mathcal{L}(tx') = -\tfrac{d}{ds}\mathcal{L}(x') = -\tfrac{d}{ds}(sX) = -(sX' + X) \\
&\mathcal{L}(tx'') = -\tfrac{d}{ds}\mathcal{L}(x'') = -\tfrac{d}{ds}(s^2X - x'(0)) = -(s^2X' + 2sX)
\end{aligned}$$

We solve

$$\begin{aligned}
&-(s^2X' + 2sX) - (sX' + X) - 2sX + X = 0 \\
&-s^2X' - 2sX - sX' - 2sX = 0 \\
&(-s^2 - s)X' = 4sX \\
&\tfrac{dX}{X} = \tfrac{4sds}{-s(s+1)} = \tfrac{-4ds}{s+1} \\
&\ln X = -4\ln|s+1| + C \\
&X = \tfrac{K}{(s+1)^4}.
\end{aligned}$$

Recall that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$; so $\mathcal{L}(t^3) = \frac{3!}{s^4}$. Recall also that if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$. It follows that $\mathcal{L}(e^{-t}t^3) = \frac{3!}{(s+1)^4}$ and

$$X = \frac{K}{3!}\mathcal{L}(e^{-t}t^3).$$

Thus, $x = \mathcal{L}^{-1}(X) = \boxed{\frac{K}{3!}(e^{-t}t^3)}$.

Check: If $x = Ae^{-t}t^3$, then

$$
\begin{aligned}
x' &= A(e^{-t}3t^2 - e^{-t}t^3) = Ae^{-t}(3t^2 - t^3)\\
x'' &= Ae^{-t}(-(3t^2 - t^3) + (6t - 3t^2)) = Ae^{-t}(6t - 6t^2 + t^3)
\end{aligned}
$$

So,

$$
\begin{aligned}
&tx'' + (t-2)x' + x\\
&= tAe^{-t}(6t - 6t^2 + t^3) + (t-2)Ae^{-t}(3t^2 - t^3) + Ae^{-t}t^3\\
&= Ae^{-t}[6t^2 - 6t^3 + t^4 + (t-2)(3t^2 - t^3) + t^3]\\
&= Ae^{-t}[6t^2 - 6t^3 + t^4 + (3t^3 - t^4 - 6t^2 + 2t^3) + t^3]\\
&= Ae^{-t}[t^4 - t^4 + (-6t^3 + 3t^3 + 2t^3 + t^3) + 6t^2 - 6t^2] = 0 \checkmark
\end{aligned}
$$