Math 242, Spring 1994, Final Exam

1. Let $\mathcal{L}(x) = X$ | $\mathcal{L}(tx) = -X'$

$\mathcal{L}(x') = sX$ | $\mathcal{L}(tx'') = -(s^2X' + 2sX)$

$\mathcal{L}(x'') = s^2X - x'(0)$ | $-s^2X' - 2sX - 2sX - X' = 0$

$\frac{dX'}{X} = \frac{-4s}{s^2+1}ds$ | $\ln X = -2\ln(s^2+1)+c$ | $X = \frac{K}{(s^2+1)^2}$

$x = \mathcal{L}^{-1}\left(\frac{K}{(s^2+1)^2}\right) = \frac{K}{2}(\sin t - t\cos t)$

$x = A(\sin t - t\cos t)$

2. $\mathcal{L}(x) = X \quad \mathcal{L}(x') = sX \quad \mathcal{L}(x'') = s^2X$

$f(t) = 1 - u(t-3)$

$X = \frac{1}{s^2+5s+4}\frac{1}{s}\left[1 - e^{-3s}\right] = \frac{1}{12}\left[\frac{3}{s} - \frac{4}{s+1} + \frac{1}{s+4}\right]\left[1-e^{-3s}\right]$

$x = \mathcal{L}^{-1}(X)$ so

$$x = \frac{1}{12}\left[3 - 4e^{-t} + e^{-4t} - u(t-3)\left(3 - 4e^{-(t-3)} + e^{-4(t-3)}\right)\right]$$

$$x = \begin{cases} \frac{1}{12}\left(3 - 4e^{-t} + e^{-4t}\right) & \text{if } t \le 3 \\ \frac{1}{12}\left(-4e^{-t}(1-e^{3}) + e^{-4t}(1-e^{12})\right) & \text{if } 3 \le t \end{cases}$$

3. Consider the initial value problem $y'' + P(x)y' + Q(x)y = f(x)$, $y'(a) = b_1$, $y(a) = b_0$. If $P(x)$, $Q(x)$ and $f(x)$ are all continuous on some interval I which contains a in its interior, then The IVP has a unique solution $y = y(x)$ which is valid for all x in I.

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④ $T(0)=30 \quad T(20)=20 \quad$ Find t_0 with $T(t_0)=5$

$\frac{dT}{dt} = -kT \qquad \ln T = -kt + c_1 \qquad T = Ce^{-kt}$

$T = 30e^{-kt} \qquad 20 = T(20) = 30e^{-k \cdot 20} \qquad \frac{\ln(\frac{2}{3})}{20} = -k$

$5 = T(t_0) = 30e^{-kt_0} \qquad \ln\frac{1}{6} = \frac{\ln(\frac{2}{3})}{20} t_0$

$t_0 = \frac{20 \ln\frac{1}{6}}{\ln(\frac{2}{3})} =$ 88.38 minutes after it was put outside

⑤ The general solution of $y''+4y'+3y=0$ is $y = c_1e^{-3x}+c_2e^{-x}$
A particular solution of $\boxed{y''+4y'+3y=16e^x}$ is $Y=2e^x$
so the general solution of $\boxed{}$ is $y = 2e^x + c_1e^{-3x} + c_2e^{-x}$

$y' = 2e^x - 3c_1e^{-3x} - c_2e^{-x}$

$3 = y(0) = 2 + c_1 + c_2$

$-3 = y'(0) = 2 - 3c_1 - c_2$

$\therefore c_1 = 2 \quad c_2 = -1 \qquad y = 2e^x + 2e^{-3x} - e^{-x}$

⑥ The general solution of $y''+25y=0$ is $y = c_1 \sin 5x + c_2 \cos 5x$
Try $y = Ax\sin 5x + Bx\cos 5x$

$y' = A[5x\cos 5x + \sin 5x] + B[-5x\sin 5x + \cos 5x]$

$y'' = A[-25x\sin 5x + 10\cos 5x] + B[-25x\cos 5x - 10\sin 5x]$

$A[-25x\sin 5x + 10\cos 5x] + B[-25x\cos 5x - 10\sin 5x]$
$+ 25(Ax\sin 5x + Bx\cos 5x) = \sin 5x$

$\therefore A = 0$ and $B = -\frac{1}{10}$

$$\boxed{y = c_1 \sin 5x + c_2 \cos 5x - \frac{1}{10} x\cos 5x}$$

(7) $x^2y'' - 4xy' + 6y = 0 \qquad r(r-1) - 4r + 6 = 0 \qquad r^2 - 5r + 6 = 0 \quad r = 2, 3$

Sol of $x^2y'' - 4xy' + 6y = 0$ is $y = c_1 x^2 + c_2 x^3$

Variation of parameters $y_{partic} = u_1 x^2 + u_2 x^3$ where

$$\begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix} \leftarrow \text{Note!} \qquad \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{x^4} \begin{bmatrix} 3x^2 & -x^3 \\ -2x & x^2 \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{x} \end{bmatrix}$$

$u_1 = -x \quad u_2 = \ln x$

$y_{partic} = -x^3 + x^3 \ln x$

$$\boxed{y = c_1 x^2 + c_2 x^3 + x^3 \ln x}$$

(8) $y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx = e^{-x} \int \frac{e^{-\int 1 - \frac{1}{x}\,dx}}{e^{-2x}}\,dx = e^{-x} \int \frac{e^{-x + \ln x}}{e^{-2x}}\,dx$

$= e^{-x} \int x e^{x}\,dx = e^{-x}\left[x e^{x} - \int e^{x}\,dx\right] = e^{-x}\left[x e^{x} - e^{x}\right]$

$= x - 1$

$$y = c_1 e^{-x} + c_2 (x-1)$$

(9) $\mu = e^{\int P(x)\,dx} = e^{\int 2 - \frac{3}{x}\,dx} = e^{2x - 3\ln x} = \frac{e^{2x}}{x^3}$

$\frac{e^{2x}}{x^3} y' + \left(\frac{2e^{2x}}{x^3} - \frac{3e^{2x}}{x^4}\right) y = 4e^{2x}$

$\frac{e^{2x}}{x^3} y = 2e^{2x} + c$

$$y = 2x^3 + c\,\frac{x^3}{e^{2x}}$$

(35)

(10) $y' = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x} \quad v = \frac{y}{x} \quad xv = y \quad xv' + v = y'$

$xv' + v = \frac{2}{v} + \frac{3}{2}v \qquad xv' = \frac{2}{v} + \frac{1}{2}v = \frac{4+v^2}{2v}$

$\frac{2v}{4+v^2}\,dv = \frac{dx}{x} \qquad \ln 4+v^2 = \ln x + k \qquad 4+v^2 = cx$

$\left(\frac{y}{x}\right)^2 = cx - 4 \qquad y = \sqrt{cx^3 - 4x^2}$