Exam 3 1993 Math 242

① $y''+2y'+y=0$ try $y=e^{rx}$ $(r^2+2r+1)=0$ $(r+1)^2=0$ $r=-1$

so gen solution of homog. problem is $y=c_1e^{-x}+c_2xe^{-x}$.

For a particular solution of the non-homogeneous problem

try $y=Ax^2e^{-x}$ $y'=A(-x^2e^{-x}+2xe^{-x})$ $y''=A(x^2e^{-x}-4xe^{-x}+2e^{-x})$

$$A(x^2e^{-x}-4xe^{-x}+2e^{-x})+2A(-x^2e^{-x}+2xe^{-x})+Ax^2e^{-x}=e^{-x}$$

$$A(2)e^{-x}=e^{-x}$$

$$\therefore A=\tfrac{1}{2}$$

$$y=c_1e^{-x}+c_2xe^{-x}+\tfrac{1}{2}x^2e^{-x}$$

$\cancel{x}^{Lb}=e^{Lb\,a\,x}=$

② Cauchy-Euler equation, try $y=x^r$

$$x^2(r(r-1)x^{r-2})+3x(rx^{r-1})+x^r=0$$

$$x^r(r^2-r+3r+1)=0$$

$$(r^2+2r+1)=0$$

$(r+1)^2=0$ $r=-1$ One solution is $y=x^{-1}$

For the other solution, we try reduction of order

$y=x^{-1}v$ $y'=x^{-1}v'-x^{-2}v$ $y''=x^{-1}v''-2x^{-2}v'+2x^{-3}v$

$$x^2\left(\frac{1}{x}v''-\frac{2}{x^2}v'+\cancel{\frac{2}{x^3}v}\right)+3x\left(\frac{1}{x}v'-\cancel{\frac{1}{x^2}v}\right)+\cancel{\frac{1}{x}v}=0$$

$$xv''+(-2+3)v'=0$$

$$x\frac{dv'}{dx}=-v'$$

$$\frac{dv'}{v'}=-\frac{1}{x}dx$$

$$\ln v'=-\ln x$$

$$v'=\frac{1}{x}$$

$$v=\ln x$$

$$y=c_1\frac{1}{x}+c_2\frac{1}{x}\ln x$$

15

③ The solution of $y''+y=0$ is $y=c_1\cos x+c_2\sin x$

A solution of $y''+y=\sec^2 x$ is $y=u_1y_1+u_2y_2$, where

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}=\begin{bmatrix} 0 \\ \sec^2 x \end{bmatrix} \qquad \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}=\begin{bmatrix} 0 \\ \sec^2 x \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}=\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}\begin{bmatrix} 0 \\ \sec^2 x \end{bmatrix}=\begin{bmatrix} -\tan x\sec x \\ \sec x \end{bmatrix}$$

15

$u_1'=-\sec x\tan x \qquad u_1=-\sec x$

$u_2'=\sec x \qquad u_2=\ln|\sec x+\tan x|$

$$y=c_1\cos x+c_2\sin x-1+\sin x\ln|\sec x+\tan x|$$

④ $F(s)=\int_0^\infty e^{-st}f(t)\,dt=\int_0^1 e^{-st}(1-t)\,dt$

$=\int_0^1 e^{-st}dt-\int_0^1 te^{-st}dt \qquad u=t \quad v=-\frac{1}{s}e^{-st}$

$du=dt \quad dv=e^{-st}dt$

$=-\frac{1}{s}e^{-st}\Big]_0^1-\left(-\frac{t}{s}e^{-st}\Big]_0^1+\frac{1}{s}\int_0^1 e^{-st}\,dt\right)$

$=-\frac{1}{s}e^{-st}+\frac{t}{s}e^{-st}-\frac{1}{s}\left(-\frac{1}{s}\right)e^{-st}\Big]_0^1$

$=\frac{1}{e^s}\left[\frac{1}{s^2}\right]-\left[-\frac{1}{s}+\frac{1}{s^2}\right]=\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^2e^s}$

5. Let $X = \mathcal{L}(x)$ $\quad \mathcal{L}(x') = s\mathcal{L}(x) - x(0) = sX$

$$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s^2 X$$

$$s^2 X + 4sX + 3X = \frac{1}{s}$$

$$X = \frac{1}{s(s+1)(s+3)}$$

10

$$\frac{1}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{s+3}$$

$$1 = A(s+1)(s+3) + B(s+3)s + C(s+1)s$$

$1 = A3 \qquad 1 = B(2)(-1) \qquad 1 = C(-2)(-3)$

$A = \frac{1}{3} \qquad B = -\frac{1}{2} \qquad C = \frac{1}{6}$

$$X = \frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}$$

$$x = \mathcal{L}^{-1}(X) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

$$x(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

17