Math 242 Fall 1994 Exam 1

PRINT Your Name:______________________

There are 7 problems on 4 pages. Problem 2 is worth 8 points. Each of the other problems is worth 7 points. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer.

1. State the Existence and Uniqueness Theorem for first order differential equations.

Consider the Initial Value Problem

(*) $\begin{cases} y' = f(x,y) \\ y(a) = b \end{cases}$.

a) If there is a rectangle R in the xy-plane, which contains (a,b) in its interior, and f is continuous on R, then (*) has at least one solution which is defined on some interval I containing $x=a$.

b) If $\frac{\partial f}{\partial y}$ is also continuous on R, then (*) has exactly one solution which is defined on some interval I containing $x=a$.

2.

(a) What does the Existence and Uniqueness Theorem tell you about the Initial Value Problem

$$(1+x^2)y' = (7+y)^2 \quad y(0) = -7?$$

(b) Solve the Initial Value Problem of part (a).

$y' = \frac{(7+y)^2}{1+x^2}$ so $f(x,y) = \frac{(7+y)^2}{1+x^2}$ which is continuous every[where]

$f_y = \frac{2(7+y)}{1+x^2}$ which is continuous everywhere.

a) Thus the IVP has a unique solution

b) $y = -7$

-3

3. Solve the Initial Value Problem $y' = x\cos x$, $y(0) = 0$.

$y = \int x\cos x\,dx = x\sin x + \cos x + C$

$0 = y(0) = \cos(0) + C$ so $C = -1$

$y = x\sin x + \cos x - 1$

4. Solve $xy' = y + 2\sqrt{xy}$.

$y' = \frac{y}{x} + 2\sqrt{\frac{y}{x}}$

Let $V = \frac{y}{x}$

$xV = y$

$xV' + V = y'$

$xV' + V = V + 2\sqrt{V}$

$\frac{dV}{\sqrt{V}} = \frac{2}{x}\,dx$

$2\sqrt{V} = 2\ln|x| + C$

$\sqrt{V} = \ln|x| + K$

$V = (\ln|x| + K)^2$

$y = x(\ln|x| + K)^2$

-2

5. Solve $y' = y + y^3$.

$V = y^{-2}$

$V' = -2y^{-3}y'$

$-2y^{-3}y' = -2y^{-2} \quad -2$

$V' + 2V = -2$

$\mu = e^{\int 2dx} = e^{2x}$

$e^{2x}v' + 2e^{2x}v = -2e^{2x}$

$e^{2x}v = -2\int e^{2x}dx$

$e^{2x}v = -e^{2x} + c$

$V = -1 + ce^{-2x}$

$y^{-2} = -1 + ce^{-2x}$

$y = \dfrac{1}{\sqrt{(-1+ce^{-2x})}}$

6. Solve the initial value problem

$$\begin{cases} y'' - 9y = 0 \\ y(0) = -1 \\ y'(0) = 9 \end{cases}$$

If it helps you, you may use the fact that $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ both are solutions of the differential equation $y'' - 9y = 0$.

$y = ae^{3x} + be^{-3x}$ $\qquad$ $y' = 3ae^{3x} - 3be^{-3x}$

$-1 = y(0) = a + b$

$9 = y'(0) = 3a - 3b$

Multiply the top equation by 3 and add

$-3 = 3a + 3b$

$9 = 3a - 3b$

$6 = 6a$

$a =$ [illegible]

$b = -2$

$y = e^{3x} - 2e^{-3x}$

-1

7. A 400 gallon tank initially contains 100 gallons of brine containing 25 pounds of salt. Brine containing 2 pounds of salt per gallon enters the tank at the rate of 5 gal./sec., and the mixed brine in the tank flows out at the rate the rate of 4 gal./sec.. How much salt will the tank contain at the moment it becomes full?

Let $X(t)$ = # of lbs of salt in the tank at time t

$X(0) = 25$

ans = $X(300)$

$$\frac{dX}{dt} = 2\,\frac{\text{lb}}{\text{gal}}\;5\,\frac{\text{gal}}{\text{sec}} - \frac{X}{100+t}\,\frac{\text{lb}}{\text{gal}}\;4\,\frac{\text{gal}}{\text{sec}}$$

$$X' + \frac{4}{100+t}X = 10$$

$$\mu = e^{\int \frac{4}{100+t}dt} = e^{4\ln 100+t} = (100+t)^4$$

$$(100+t)^4 X' + 4(100+t)^3 X = 10(100+t)^4$$

$$(100+t)^4 X = 2(100+t)^5 + C$$

$$X(t) = 2(100+t) + \frac{C}{(100+t)^4}$$

$$25 = X(0) = 200 + \frac{C}{(100)^4}$$

$$-175(100)^4 = C$$

$$\text{ans} = X(300) = 2(400) - \frac{175(100)^4}{(400)^4} = 800 - \frac{175}{256} = 799.316 \text{ lb}$$

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