# Math 242, Exam 2, Spring 2012

Write everything on the blank paper provided.

**You should KEEP this piece of paper.**

If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **5** problems. Each problem is worth 10 points.

SHOW your work. $\boxed{\mathit{CIRCLE}}$ your answer. Write **coherently.**

**No Calculators or Cell phones.**

I will post the solutions later today.

1. (8 points) **Solve** $xy\frac{dy}{dx} = y^2 + x\sqrt{4x^2+y^2}$**. Express your answer in the form** $y$ **is some function of** $x$**. Check your answer.**

Divide by $x^2$. The problem maybe solved using the homogeneous substitution $v = y/x$. We have $xv = y$ so $x\frac{dv}{dx} + v = \frac{dy}{dx}$. The original problem is

$$\frac{y}{x}\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \sqrt{4 + \left(\frac{y}{x}\right)^2}$$

$$v(x\tfrac{dv}{dx} + v) = v^2 + \sqrt{4+v^2}$$

$$vx\frac{dv}{dx} = \sqrt{4+v^2}$$

$$\frac{v}{\sqrt{4+v^2}}dv = \frac{1}{x}dx$$

$$\sqrt{4+v^2} = \ln x + C$$

$$4 + v^2 = (\ln x + C)^2$$

$$v^2 = (\ln x + C)^2 - 4$$

$$v = \pm\sqrt{(\ln x + C)^2 - 4}$$

But $v = y/x$; so,

$$\boxed{y = \pm x\sqrt{(\ln x + C)^2 - 4}.}$$

**Check.** We plug $y = x\sqrt{(\ln x + C)^2 - 4}$, with $2 \le \ln x + C$, into the original DE and see if it works. We see that

$$xy\frac{dy}{dx} = x(x\sqrt{(\ln x + C)^2 - 4})\left[x\frac{2(\ln x + C)}{2x\sqrt{(\ln x + C)^2 - 4}} + \sqrt{(\ln x + C)^2 - 4}\right]$$

$$= x^2\sqrt{(\ln x + C)^2 - 4}\left[\frac{(\ln x + C)}{\sqrt{(\ln x + C)^2 - 4}} + \sqrt{(\ln x + C)^2 - 4}\right]$$

$$= x^2(\ln x + C) + x^2((\ln x + C)^2 - 4).$$

On the other hand,

$$y^2 + x\sqrt{4x^2 + y^2} = \left(x\sqrt{(\ln x + C)^2 - 4}\right)^2 + x\sqrt{4x^2 + \left(x\sqrt{(\ln x + C)^2 - 4}\right)^2}$$

$$= x^2[(\ln x + C)^2 - 4] + x\sqrt{4x^2 + x^2[(\ln x + C)^2 - 4]}$$

$$= x^2[(\ln x + C)^2 - 4] + x\sqrt{x^2(\ln x + C)^2}$$

$$= x^2[(\ln x + C)^2 - 4] + x^2(\ln x + C).$$

These two expressions are equal. Our proposed solution is correct.

2. (7 points) **Solve** $x^2\frac{dy}{dx} + 2xy = 5y^4$. **Express your answer in the form** $y$ **is some function of** $x$. **Check your answer.**

This is a Bernoulli equation. Let $v = y^{-3}$. It follows that $\frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$. Divide the given Differential Equation by $y^4$ to obtain

$$x^2y^{-4}\frac{dy}{dx} + 2xy^{-3} = 5.$$

Substitute to get

$$\frac{x^2}{-3}\frac{dv}{dx} + 2xv = 5.$$

Multiply by $-\frac{3}{x^2}$ to get

$$\frac{dv}{dx} - \frac{6}{x}v = \frac{-15}{x^2}.$$

The Differential Equation now has the form $\frac{dv}{dx} + P(x)v = Q(x)$; this is a first order linear differential equation. Let $\mu = e^{\int P(x)dx} = e^{\int \frac{-6}{x}dx} = e^{-6\ln x} = x^{-6}$. Multiply both sides of the equation by $x^{-6}$:

$$x^{-6}\frac{dv}{dx} - 6x^{-7}v = -15x^{-8}.$$

Thus, $\frac{d(x^{-6}v)}{dx} = -15x^{-8}$ and $x^{-6}v = \frac{15}{7}x^{-7} + C$. We have $v = \frac{15}{7x} + Cx^6$. But $v = y^{-3}$; so, $\boxed{y = (\frac{15}{7}x^{-1} + Cx^6)^{-1/3}}$.

**Check.** We plug our proposed answer into the original DE and see if it works. We see that

$$x^2\frac{dy}{dx} + 2xy = x^2(-1/3)(\tfrac{15}{7}x^{-1} + Cx^6)^{-4/3}(-\tfrac{15}{7}x^{-2} + 6Cx^5) + 2x(\tfrac{15}{7}x^{-1} + Cx^6)^{-1/3}$$

$$= (\tfrac{15}{7}x^{-1} + Cx^6)^{-4/3}\left[x^2(-1/3)(-\tfrac{15}{7}x^{-2} + 6Cx^5) + 2x(\tfrac{15}{7}x^{-1} + Cx^6)\right]$$

$$= (\tfrac{15}{7}x^{-1} + Cx^6)^{-4/3}\left[\tfrac{5}{7} - 2Cx^7 + \tfrac{30}{7} + 2Cx^7\right] = 5(\tfrac{15}{7}x^{-1} + Cx^6)^{-4/3}$$

$$= 5y^4. \checkmark$$

3. (7 points) **State the Existence and Uniqueness Theorem for first order differential equations.**

Consider the Initial Value Problem IVP: $y' = f(x, y)$ with $y(x_0) = y_0$.

(a) If $f$ is continuous on some rectangle that contains $(x_0, y_0)$ in its interior, then IVP has a solution on some interval containing $x_0$.

(b) If $f$ and $f_y$ are both continuous on some rectangle that contains $(x_0, y_0)$ in its interior, then IVP has a unique solution on some interval containing $x_0$.

4. (7 points) **A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. A salt water solution which contains 2 kg of salt in each liter of solution is pumped into the tank at the rate of 5 L/s, and the mixture — kept uniform by stirring — is pumped out at the same rate. How long will it be until there are 500 kg of salt in the tank?**

Let $x(t)$ be the number of kg of salt in the tank at time t seconds. The problem tells us that $\frac{dx}{dt} = \frac{5L}{s}\frac{2\text{kg}}{L} - \frac{5L}{s}\frac{x\text{kg}}{1000L}$. So, $\frac{dx}{dt} = 10 - \frac{x}{200}$. Separate the variables:

$$\frac{dx}{dt} = \frac{2000 - x}{200}$$

$$\frac{dx}{2000 - x} = \frac{dt}{200}$$

Integrate to see

$$-\ln|2000 - x| = \frac{1}{200}t + C$$

$$\ln|2000 - x| = -\frac{1}{200}t - C$$

$$2000 - x = Ke^{-\frac{1}{200}t}$$

(where $K = \pm e^{-C}$ )

$$2000 - Ke^{-\frac{1}{200}t} = x(t)$$

The problem tells us that $100 = x(0) = 2000 - K$ ; so, $x(t) = 2000 - 1900e^{-\frac{1}{200}t}$ . We are supposed to find the time when $500 = x(t)$ . So we solve for $t$ : $500 = 2000 - 1900e^{-\frac{1}{200}t}$ which becomes

$$1900e^{-\frac{1}{200}t} = 1500$$

$$e^{-\frac{1}{200}t} = \frac{1500}{1900}$$

$$-\frac{1}{200}t = \ln(\tfrac{15}{19})$$

$$\frac{1}{200}t = \ln(\tfrac{19}{15})$$

$$\boxed{t = 200\ln(\tfrac{19}{15}) \text{ seconds}}.$$

5. (7 points) **When the brakes are applied to a certain car, the acceleration of the car is $-k$ m/s$^2$ for some positive constant $k$. Suppose that the car is traveling at the velocity $v_0$ m/s when the brakes are first applied and that the brakes continue to be applied until the car stops.**

 (a) **Find the distance that the car travels between the moment that the brakes are first applied and the moment when the car stops. (Your answer will be expressed in terms of $k$ and $v_0$ .)**

 Let $x(t)$ be the position of the car at time $t$ . We take $t = 0$ to be the moment that the brakes are applied. So $v(0) = v_0$ and $x(0) = 0$ . We are told $x'' = -k$ . We integrate and plug in the points to see $v(t) = -kt + v_0$ and $x(t) = -kt^2/2 + v_0 t$ . Let $t_s$ be the time when the car stops. We have $0 = v(t_s) = -kt_s + v_0$ . Thus, $t_s = v_0/k$ . The distance traveled while the brakes were applied is

$$x(t_s) = x(v_0/k) = -k(v_0/k)^2/2 + v_0(v_0/k) = (v_0^2/k)(1 - 1/2) = \boxed{\frac{v_0^2}{2k}}.$$

 (b) **How does the stopping distance change if the initial velocity is changed to $4v_0$ ?**

 $\boxed{\text{The stopping distance is multiplied by } 4^2}$ if $v_0$ is replaced by $4v_0$ .

6. (7 points) **Consider the Initial Value Problem** $\frac{dx}{dt} = (x-1)(x-3)$, $x(0) = x_0$.

   (a) **Solve the Initial Value Problem.**
   (b) **Draw some of the solutions.**
   (c) **Which choices for** $x_0$ **cause** $x$ **to go to infinity at some finite time?**
   (d) **Which choices for** $x_0$ **cause** $x$ **to go toward a finite constant as** $t$ **goes to infinity.**

We see that if

$$\frac{1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3},$$

then $1 = A(x-3) + B(x-1)$. Plug in $x = 1$ to see that $A = -1/2$. Plug in $x = 3$ to see $B = 1/2$. We check that

$$\frac{1}{2}\left(\frac{1}{x-3} - \frac{1}{x-1}\right) = \frac{1}{(x-1)(x-3)}.$$

We solve the differential equation:

$$\int \frac{dx}{(x-1)(x-3)} = \int dt$$

$$\frac{1}{2}\int\left(\frac{1}{x-3} - \frac{1}{x-1}\right)dx = \int dt$$

$$\frac{1}{2}\left[\ln|x-3| - \ln|x-1|\right] = t + C$$

$$\left[\ln|x-3| - \ln|x-1|\right] = 2t + 2C$$

$$\left|\frac{x-3}{x-1}\right| = e^{2C}e^{2t}$$

$$\frac{x-3}{x-1} = \pm e^{2C}e^{2t}$$

Let $K$ be the constant $\pm e^{2C}$; so,

$$\frac{x-3}{x-1} = Ke^{2t}.$$

Plug in $t = 0$ to see that $K = \frac{x_0 - 3}{x_0 - 1}$. Multiply both sides by $x - 1$ to see

$$x - 3 = Ke^{2t}(x-1).$$

Add $-Ke^{2t}x$ to both sides to obtain

$$x - Ke^{2t}x = 3 - Ke^{2t}.$$

So,

$$x = \frac{3 - Ke^{2t}}{1 - Ke^{2t}}$$

$$x = \frac{3 - \left(\frac{x_0-3}{x_0-1}\right)e^{2t}}{1 - \left(\frac{x_0-3}{x_0-1}\right)e^{2t}}$$

$$\boxed{x(t) = \frac{3(x_0-1) - (x_0-3)e^{2t}}{(x_0-1) - (x_0-3)e^{2t}}} \quad \text{answer to (a).}$$

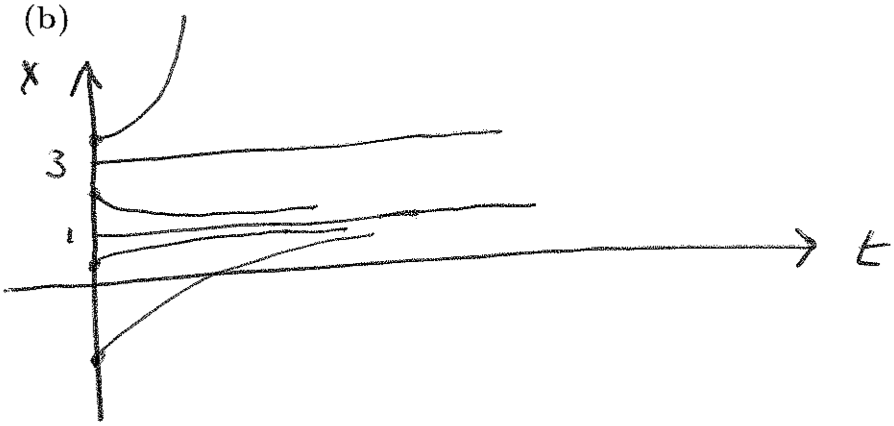
**(c)** If $3 < x_0$, then the denominator of $x(t)$ becomes zero at the positive time $t = (1/2)\ln\left(\frac{x_0-1}{x_0-3}\right)$. In other words, if $3 < x_0$, then $x(t)$ goes to infinity in finite time.

**(d)** If $x_0 < 3$, then the denominator of $x(t)$ is positive for all non-negative $t$. In this case,

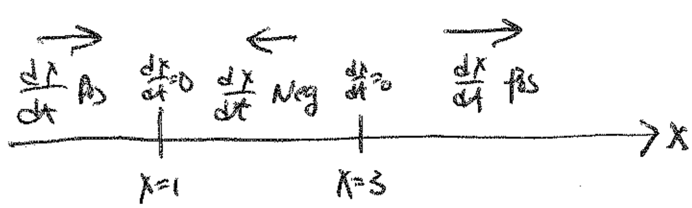
$$\lim_{x\to\infty} x(t) = \lim_{x\to\infty} \frac{3(x_0-1) - (x_0-3)e^{2t}}{(x_0-1) - (x_0-3)e^{2t}} = \lim_{x\to\infty} \frac{3(x_0-1)e^{-2t} - (x_0-3)}{(x_0-1)e^{-2t} - (x_0-3)}$$

$$= \frac{x_0-3}{x_0-3} = 1.$$

**(b)**

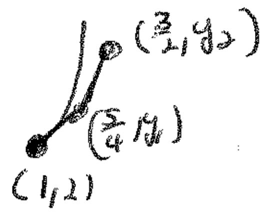



**Note.** Of course the phase diagram gives much of the qualitative information about our answer to problem 1.



7. (7 points) **Consider the initial value problem $\frac{dy}{dx} = x + y^2$, $y(1) = 2$. Use Euler's method to approximate $y(3/2)$. Use two steps, each of size $1/4$.**

Consider the picture:




The curve represents the correct solution of the initial value problem. We approximate the real solution with the two line segments. The line segment from $(1,2)$ to $(\frac{5}{4}, y_1)$ has slope equal to $(x+y^2)|_{(1,2)}$. The line segment from $(\frac{5}{4}, y_1)$ to $(\frac{3}{2}, y_2)$ has slope equal to $(x+y^2)|_{(\frac{5}{4}, y_1)}$. Of course, $y_2$ is our approximation of $y(\frac{3}{2})$.

We see that $(x+y^2)|_{(1,2)} = 5$; so, the segment from $(1,2)$ to $(\frac{5}{4}, y_1)$ has slope equal to $5$. This segment lives on the line

$$y - 2 = 5(x-1),$$

which is $y = 5x - 3$. Thus, $y_1 = 25/4 - 3 = 13/4$.

The line segment from $(\frac{5}{4}, y_1)$ to $(3/2, y_2)$ has slope equal to

$$(x+y^2)|_{(\frac{5}{4}, y_1)} = (x+y^2)|_{(\frac{5}{4}, \frac{13}{4})} = \frac{5}{4} + \left(\frac{13}{4}\right)^2 = \frac{189}{16}.$$

This segment lives on the line

$$y - \frac{13}{4} = \frac{189}{16}\left(x - \frac{5}{4}\right)$$

which is

$$y = \frac{189}{16}x - \frac{5(189)}{64} + \frac{13}{4}$$

Our approximation of $y(3/2)$ is

$$\boxed{y_2 = \frac{189}{16}\left(\frac{3}{2}\right) - \frac{5(189)}{64} + \frac{13}{4}.}$$