Please PRINT your name _____

No calculators, cell phones, computers, notes, etc.

Circle your answer. Make your work correct, complete and coherent.

The quiz is worth 5 points. The solutions will be posted on my website later today.

Quiz 3, February 1, 2021

Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/sec^2, while air resistance provides $\frac{1}{10}$ ft/sec^2 of deceleration for each foot per second of the car's velocity.

(a) Find $\lim_{t\to\infty} v(t)$, where $v(t)$ is the velocity of the car at time t.

(b) How long does it take for the car to reach 90% of the answer to (a)? How far does it travel while doing so?

ANSWER: We must solve the Initial Value Problem

$$\frac{dv}{dt} = 10 - \tfrac{1}{10}v \quad v(0) = 0.$$

Separate the variables and integrate

$$\int \frac{dv}{10 - \frac{1}{10}v} = \int dt$$

$$-10\ln|10 - \tfrac{1}{10}v| + C = t$$

When $t = 0$, then $v = 0$; so

$$-10\ln 10 + C = 0$$

and $C = 10\ln 10$. Thus,

$$-10\ln|10 - \tfrac{1}{10}v| = t - 10\ln 10.$$

Divide both sides by -10 to obtain

$$\ln|10 - \tfrac{1}{10}v| = \tfrac{-t}{10} + \ln 10.$$

Exponentiate both sides to obtain

$$|10 - \tfrac{1}{10}v| = 10e^{-t/10}.$$

At the beginning of the problem, $v = 0$ hence at least at the beginning of the problem $10 - \frac{1}{10}v$ is positive and

$$|10 - \tfrac{1}{10}v| = 10 - \tfrac{1}{10}v.$$

Thus,

$$10 - \tfrac{1}{10}v = 10e^{-t/10}$$

$$10 - 10e^{-t/10} = \tfrac{1}{10}v$$

Multiply both sides by 10

$$100 - 100e^{-t/10} = v.$$

(We have solved the IVP. Now we can answer the questions.) Observe that

$$\lim_{t\to\infty} v = \lim_{t\to\infty}(100 - 100e^{-t/10}) = 100$$

$$\boxed{\text{The limit of the car's velocity is 100 feet/sec.}}$$

The car reaches the speed 90, when $v(t) = 90$; so

$$\begin{aligned} 100 - 100e^{-t/10} &= 90 \\ 100 - 90 &= 100e^{-t/10} \\ 10 &= 100e^{-t/10} \\ \tfrac{1}{10} &= e^{-t/10} \\ \ln\tfrac{1}{10} &= -t/10 \\ -10\ln\tfrac{1}{10} &= t \end{aligned}$$

Of course $-\ln x = \ln\frac{1}{x}$.

$$\boxed{\text{The car reaches the velocity 90 feet/second after } 10\ln 10 \text{ seconds.}}$$

As the car travels from rest to 90 feet/second, the ditance it travels is

$$\int_0^{10\ln 10} v(t)dt = \int_0^{10\ln 10} (100 - 100e^{-t/10})dt$$

$$= 100t + 1000e^{-t/10}\Big|_0^{10\ln 10} = 1000\ln 10 + 1000e^{-\ln 10} - 1000$$

$$= 1000\ln 10 + 100 - 1000 = \boxed{(1000\ln 10 - 900)\text{feet}}$$

Of course, we used $e^{-\ln 10} = \frac{1}{10}$.