Quiz 2, January 24, 2017, 1:15 class

Suppose that a car starts from rest, its engine providing an acceleration of 10 feet/second^2, while air resistance provides $1/10$ feet/second^2 of deceleration for each foot per second of the car's velocity.

(a) Find the car's maximum possible (limiting) velocity.

(b) Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.

ANSWER: Let $v(t)$ be the velocity of the car at time t, where distance is measured in feet and time in seconds. We are interested in the Initial Value Problem:

$$\begin{cases} \frac{dv}{dt} = 10 - \frac{1}{10}v \\ v(0) = 0. \end{cases}$$

We first find the formula for v as a function of t. Separate the variables and integrate:

$$\int \frac{dv}{10-\frac{1}{10}v} = \int dt$$

$$-10\ln|10 - \tfrac{1}{10}v| = t + C$$

$$\ln|10 - \tfrac{1}{10}v| = \tfrac{t}{-10} + \tfrac{C}{-10}$$

$$|10 - \tfrac{1}{10}v| = e^{\frac{C}{-10}}e^{\frac{t}{-10}}$$

$$10 - \tfrac{1}{10}v = \pm e^{\frac{C}{-10}}e^{\frac{t}{-10}}$$

Let $K = \pm e^{\frac{C}{-10}}$.

$$10 - \tfrac{1}{10}v = Ke^{\frac{t}{-10}}$$

$$10 - Ke^{\frac{t}{-10}} = \tfrac{1}{10}v$$

When $t = 0$, then $v = 0$; so, $10 - K = 0$ and $K = 10$.

$$10 - 10e^{\frac{t}{-10}} = \tfrac{1}{10}v$$

Multiply both sides by 10

$$100(1 - e^{\frac{t}{-10}}) = v.$$

We have finally accomplished our first goal.

The answer to (a) is $\lim_{t\to\infty} v = \lim_{t\to\infty} 100(1 - e^{\frac{t}{-10}}) = \boxed{100\,\text{feet/second}}$

Now we do (b). The car reaches 90 feet/second, when

$$100(1 - e^{\frac{t}{-10}}) = 90$$

$$1 - e^{\frac{t}{-10}} = \tfrac{9}{10}$$

$$1 - \tfrac{9}{10} = e^{\frac{t}{-10}}$$

$$\ln(\tfrac{1}{10}) = \tfrac{t}{-10}$$

$$10\ln 10 \text{ seconds} = t$$

We used $\ln\frac{1}{10} = -\ln 10$.

The position of the car at time t is

$$x(t) = \int v(t)dt = \int 100(1 - e^{\frac{t}{-10}})dt$$
$$= 100(t + 10e^{\frac{t}{-10}}) + c_2$$

If $x(0) = 0$, then $0 = 1000 + c_2$ and $c_2 = -1000$. Thus, the position of the car at time t is

$$x(t) = 100(t + 10e^{\frac{t}{-10}}) - 1000$$

The position of the car at time $10\ln 10$ seconds is

$$x(10\ln 10) = 100(10\ln 10 + 10e^{\frac{10\ln 10}{-10}}) - 1000$$
$$= 1000\ln 10 + \frac{1000}{10} - 1000 = 1000\ln 10 - 900.$$

The maximum velocity of the car is 100 f/s. The car reaches the velocity of 90 f/s after $10\ln 10$ seconds. The car has traveled $1000\ln 10 - 900$ feet before it reaches the velocity of 90 f/s.