Math 242, Exam 3, Spring, 2018 Solutions

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today. The exam will be returned in class on Thursday.

No Calculators or Cell phones.

(1) **Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/sec^2, while air resistance provides 1/10 ft/sec^2 of deceleration for each foot per second of the car's velocity. Find the car's maximum possible (that is, limiting) velocity.**

Let $v(t)$ be the velocity of the car at time t. We are told that $v'(t) = 10-(1/10)v$. We want $\lim\limits_{t\to\infty} v(t)$. Separate the variables and integrate:

$$\int \frac{dv}{10-(1/10)v} = \int dt$$

$$\int \frac{10dv}{100-v} = \int dt$$

$$-10\ln(100-v) = t + C$$

$$\ln(100-v) = -(1/10)(t+C)$$

$$100 - v = Ke^{-(1/10)t} \quad \text{where } K = e^{-(1/10)C}$$

$$100 - Ke^{-(1/10)t} = v(t)$$

$$\boxed{\lim_{t\to\infty} v(t) = 100 \text{ ft/sec}}$$

because $\lim\limits_{t\to\infty} -Ke^{-(1/10)t} = 0$.

(2) **Solve** $4y''' + 12y'' + 9y' = 0$.

This is a third order homogeneous linear differential equation with contant coefficients. We try $y = e^{rx}$. We consider the characteristic equation

$$4r^3 + 12r^2 + 9r = 0$$

$$r(4r^2 + 12r + 9) = 0$$

$$r(2r+3)^2 = 0$$

The answer is

$$\boxed{y = c_1 + c_2e^{-(3/2)x} + c_3xe^{-(3/2)x}.}$$

(3) **Solve the Initial Value problem $y'' + 9y = \cos 2x$, $y(0) = 1$, $y'(0) = 0$.**

The solution of the homogeneous problem is $y = c_1 \sin 3x + c_2 \cos 3x$. To find a particular solution of the given problem we try $y = A\cos 2x$ because $y'' = -4A\cos 2x$ (which does not involve $\sin 2x$). We look for A with

$$-4A\cos 2x + 9A\cos 2x = \cos 2x.$$

We take $A = 1/5$. The general solution of the given DE is

$$y = c_1 \sin 3x + c_2 \cos 3x + (1/5)\cos 2x.$$

We must find c_1 and c_2 with $y(0) = 1$ and $y'(0) = 0$.

$$y' = 3c_1 \cos 3x - 3c_2 \sin 3x - (2/5)\sin 2x.$$

We see that

$$1 = y(0) = c_2 + (1/5),$$
$$0 = y'(0) = 3c_1.$$

We conclude that

$$\boxed{y(x) = (4/5)\cos 3x + (1/5)\cos 2x.}$$

Check. We calculate

$$y'(x) = (-12/5)\sin 3x - (2/5)\sin 2x$$

$$y''(x) = (-36/5)\cos 3x - (4/5)\cos 2x$$

Observe that

$$y'' + 9y = \cos 2x, \quad y(0) = 1, \quad y'(0) = 0.$$

(4) **Solve $y' + \frac{4}{x}y = x^3y^2$.**

This is a Bernoulli equation. Let $v = y^{-1}$. Observe that $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. Multiply both sides by $-y^{-2}$:

$$-y^{-2}y' + \frac{-4}{x}y^{-1} = -x^3$$
$$\frac{dv}{dx} + \frac{-4}{x}v = -x^3$$

Multiply both sides by

$$e^{\int P(x)dx} = e^{\int \frac{-4}{x}dx} = e^{-4\ln x} = x^{-4}$$

to obtain

$$x^{-4}\frac{dv}{dx} + -4x^{-5}v = -x^{-1}.$$

Observe that the left side is

$$\frac{d}{dx}(x^{-4}v).$$

Integrate both sides with respect to x to obtain

$$x^{-4}v = -\ln|x| + C$$
$$v = -x^4\ln|x| + Cx^4$$
$$\frac{1}{y} = -x^4\ln|x| + Cx^4$$
$$\boxed{\frac{1}{-x^4\ln|x| + Cx^4} = y.}$$

Check. We check $\frac{1}{-x^4\ln x + Cx^4} = y$ We calculate

$$y' + \frac{4}{x}y = -(-x^4\ln x + Cx^4)^{-2}(-x^4(1/x) - 4x^3\ln x + 4x^3C) + \frac{4}{x}(-x^4\ln x + Cx^4)^{-1}$$
$$= (-x^4\ln x + Cx^4)^{-2}[x^3 + 4x^3\ln x - 4x^3C + \frac{4}{x}(-x^4\ln x + Cx^4)]$$
$$(-x^4\ln x + Cx^4)^{-2}x^3 = y^2x^3.$$

(5) **Solve the Initial Value Problem** $\frac{dx}{dt} = x^2 - 5x + 4$**,** $x(0) = x_0$**.**

Separate the variables and integrate

$$\int \frac{dx}{x^2 - 5x + 4} = \int dt.$$

Observe that $x^2 - 5x + 4 = (x-4)(x-1)$. If

$$\frac{1}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1},$$

then

$$1 = A(x-1) + B(x-4).$$

Plug in $x = 1$ to see that $B = -(1/3)$. Plug in $x = 4$ to see that $A = 1/3$.

$$(1/3)\int \frac{1}{x-4} - \frac{1}{x-1}dt = \int t$$
$$(1/3)\ln\left|\frac{x-4}{x-1}\right| = t + C$$
$$\frac{x-4}{x-1} = Ke^{3t},$$

where $\pm K = e^{3C}$. This is a good place to see that

$$\frac{x_0 - 4}{x_0 - 1} = K.$$

$$x - 4 = Ke^{3t}(x-1)$$
$$x(1 - Ke^{3t}) = 4 - Ke^{3t}$$

$$x(t) = \frac{4 - Ke^{3t}}{1 - Ke^{3t}}$$

$$x(t) = \frac{4 - \frac{x_0-4}{x_0-1}e^{3t}}{1 - \frac{x_0-4}{x_0-1}e^{3t}}$$

$$\boxed{x(t) = \frac{4(x_0 - 1) - (x_0 - 4)e^{3t}}{(x_0 - 1) - (x_0 - 4)e^{3t}}.}$$